Hidden nonlinearity in a topological model of electromagnetism

Antonio F. Rañada

Departamento de Electricidad y Electrónica,
Universidad Complutense, 28040 Madrid, Spain

ABSTRACT. A subtle form of nonlinearity is described. It is shown that it appears in a topological model of electromagnetism, where it explains why the electromagnetic helicity and the charge (both electric and magnetic) can be understood as topological constants of the motion.

1 Two ways of linearising

Nonlinear field equations are frequently linearised. This is usually performed in two different ways: by truncation or by change of variables. In the first method, the second and higher order terms are neglected in the Taylor expansion of the equation. In the second one, the old variables $u_\alpha^{old}(\mathbf{r},t)$ are changed to new variables $v_\beta^{new}(\mathbf{r},t)$ by means of the equations

$$v_\beta^{new} = G_\beta(u_\alpha^{old}, \partial_\mu u_\alpha^{old}). \tag{1}$$

The first method is useful only when dealing with weak fields u_α^{old}, while the second one is possible, it has a much wider validity. In the latter case, the application to a particular nonlinear field equation would seem to suggest that the nonlinearity of the first equation is no more than an accident due to a particularly unfortunate election of the field variables, so that a completely linear theory is possible, with all the properties of the linearity.

This is certainly true if the following condition is verified: that the change of variables (1) is invertible so that the inverse change is well defined for any v_β^{new} as

$$u_\alpha^{old} = H_\alpha(v_\beta^{new}, \partial_\mu v_\beta^{new}), \tag{2}$$

which establishes a one to one correspondence between the solutions of the old and the new equations. In other words, if the application of (1) to the set of the solutions of the nonlinear equations gives the complete set of the solutions of of the linear one, the converse being also true.

We will consider in this paper an example in which this property does not hold. More precisely, in which the change (1) does not give all the solutions of the linear equation while effectively linearising the nonlinear one. This implies that the theory can be formally presented in a linear language, but not all the linear combinations of solutions are themselves solutions. In other words, although the theory looks linear because all its solutions verify a linear equation, it retains some hidden form of nonlinearity.

This case is the Maxwell's theory in empty space or with point charges. The electromagnetic field $F_{\mu\nu}$ is writen in terms of a couple of scalar fields ϕ, θ which obey highly nonlinear equations, but there are linear combinations of solutions of the Maxwell's equations that are not solutions of these equations. However, both equations, the linear and the nonlinear equations are locally equivalent, their difference being just on the behaviour at infinity.

2 Why a topological theory of electromagnetism

During a great part of the 19th century, electromagnetism was conceived in terms of lines of force, which were thought to be very real, not just a mathematical convenience. One reason for this understanding was, of course, the belief in the existence of the aether, which offered an appealing possibility to explain electromagnetic phenomena: the force lines were a manifestation of its streamlines and vorticity lines. It was expected, therefore, that the electromagnetism would be eventually understood thanks to the mechanics of fluids, in a model in which the force lines would coincide with lines of aether particles and would be therefore something real and tangible [1]. Maxwell himself was very much in favour of Kelvin's suggestion in 1868 that atoms were knots or links of the vortex lines of the ether, a picture presented expressively in a paper called "On vortex atoms" [2, 3, 4]. He liked the idea, as it expressed for instance in his presentation of the term "Atomism" in the Encyclopaedia Britannica in 1875 [5, 6].

Kelvin had applied to his topological idea the then new Helmholtz's theorems on fluid dynamics. He did not like the then widely held view of infinitely hard point atoms or, in his own words, "the monstrous assumption of infinitely strong and infinitely rigid pieces of matter" [7]. Kelvin was much impressed by the conservation of the strength of the vorticity tubes in an inviscid fluid according to Helmholtz's theorems, thinking that this was an inalterable quality on which to base an atomic theory of matter without infinitely rigid entities. We know now that this is also a trait of topological models, in which some invariant numbers characterize configurations which are rigid and can deform, distort or warp. As he put it "Helmholtz has proved an absolutely unalterable quality in any motion of a perfect liquid ... any portion [of it] has one recommendation of Lucretius' atoms — infinitely perennial specific quality".

Kelvin had the insight that such knots and links would be extremely stable, just as matter is. Furthermore, he thought that the variety of the properties of the chemical elements could be a consequence of the many different ways in which such curves can be linked or knotted. Two other important properties of matter, not known in his time [8], can also be understood. One is the ability of atoms to change into another kind in a nuclear reaction, which could be related to the breaking and reconnection of lines, as happens for instance to the magnetic lines in tokamak and astrophysical plasmas. The other is the discrete character of the spectrum, which is also a property of the nontrivial topological configurations of a vector field, as was shown by Moffatt [9].

The reception to Kelvin's idea was good, but unfortunately it was soon forgotten. Ironically, this was mainly due to Maxwell's monumental *Treatise on Electromagnetism*, after which, because of the successful developments of algebra and differential geometry, the line of force was relegated behind the concepts of electromagnetic tensor $F_{\mu\nu}$ and electromagnetic vectors E_i, B_j, A_μ. It is usually now a secondary concept, always derived from $F_{\mu\nu}$ as the integral lines of $\mathbf{B}$ and $\mathbf{E}$.

It turns out that a topological theory of electromagnetism is possible. The rest of this paper contains a terse summary.

3 Force lines and topology

Let us now try to describe the dynamics of the electromagnetic field by the evolution of its magnetic and electric lines or, in other words, let us attempt a line-dynamics (For the time being, we consider only the case of empty space; point charges can be introduced later.) As a simple tentative idea, we may represent the magnetic lines by the equation $\phi(t,x,y,z) = \phi_0$ where ϕ is a complex function of space and time and ϕ_0 is a constant labeling each line. This means that the magnetic lines are the level curves of $\phi(t,x,y,z)$. As the magnetic field is tangent to them, it can always be written as $\mathbf{B} = g(\phi,\bar\phi)\boldsymbol{\nabla}\bar\phi \times \boldsymbol{\nabla}\phi$, because of the condition $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$ (bars over complex numbers indicating complex conjugation). This can also be written as

$$B_k = -\frac{1}{2}\epsilon_{ijk}F_{ij}, \tag{3}$$

where

$$F_{\mu\nu} = -g(\phi,\bar\phi)(\partial_\mu\bar\phi\partial_\nu\phi - \partial_\nu\bar\phi\partial_\mu\phi), \tag{4}$$

the electric field being $E_i = F_{0i}$, or $\mathbf{E} = -g(\phi,\bar\phi)(\partial_0\bar\phi\boldsymbol{\nabla}\phi - \partial_0\phi\boldsymbol{\nabla}\bar\phi)$.

As we see, an antisymmetric rank 2 tensor appears, which is similar to the Faraday tensor. It turns out that $\mathbf{E}\cdot\mathbf{B} = 0$ or, equivalently, $\det(F_{\mu\nu}) = 0$, the electric and the vector fields being orthogonal, which means that, with this method, the Faraday 2-form is degenerate and the field is of radiation type.

We will admit that the total energy is finite, which implies of course that $\mathbf{B}$ and $\mathbf{E}$ go to zero at infinity. The simplest way for this condition to be achieved this condition is requiring that the limit of ϕ when $r\to\infty$ does not depend on the direction or, stated otherwise, that ϕ takes only one value at infinity.

This argument has an important consequence. To take ϕ one-valued at infinity implies that R^3 is, in fact, compactified to S^3 and that $\phi(\mathbf{r},t)$ can be interpreted at any time as a map $S^3 \mapsto S^2$, after identifying, via stereographic projection, $R^3\cup\{\infty\}$ with S^3 and the complete complex plane C with S^2. Maps of this kind have nontrivial topological properties, so that *the attempt to describe electromagnetism by the evolution of the magnetic lines, represented as the level curves of a complex function, leads in a compelling and almost unavoidable way to the appearance of a topological structure.* And a very rich one, as it happens.

Let us consider now the normalized area 2-form σ in the sphere S^2. Its pull-back to the $S^3\times R$ (identified with the spacetime) is

$$\phi^*\sigma = \frac{1}{2\pi i}\,\frac{d\phi\wedge d\bar\phi}{(1+\bar\phi\phi)^2}. \tag{5}$$

(We take the complex number $\phi(\mathbf{r}, t)$ as a coordinate in S^2.) As we see, there is a 2-form closely associated to the scalar ϕ, the level curves of which coincide with the magnetic lines. Since both $\phi^*\sigma$ and the Faraday 2-form $\mathcal{F} = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$ are closed, it seems natural to identify the two, up to a normalization constant factor which, for later convenience, we write as $-\sqrt{a}$. More precisely, we assume that

$$\mathcal{F} = -\sqrt{a}\, \phi^* \sigma, \tag{6}$$

and, consequently,

$$F_{\mu\nu} = \frac{\sqrt{a}}{2\pi i} \frac{\partial_\mu \bar{\phi} \partial_\nu \phi - \partial_\nu \bar{\phi} \partial_\mu \phi}{(1 + \bar{\phi}\phi)^2}. \tag{7}$$

Note that a is a normalizing constant which takes care that $F_{\mu\nu}$ will have the right dimensions. In natural units, a is a pure number; in MKS physical units it is an action times velocity.

As long as no charges are present, we can play the same game with the electric field $\mathbf{E}$ and a scalar field θ, the level curves of which coincide with the electric lines. In that case, if the pull-back of the area 2-form in S^2 by θ is

$$\theta^* \sigma = \frac{1}{2\pi i} \frac{d\theta \wedge d\bar{\theta}}{(1 + \bar{\theta}\theta)^2}, \tag{8}$$

and the dual to the Faraday form is taken to be

$$*\mathcal{F} = \sqrt{a}\, \theta^* \sigma. \tag{9}$$

The dual to the Faraday tensor is then

$$^*F_{\mu\nu} = \frac{\sqrt{a}}{2\pi i} \frac{\partial_\mu \theta \partial_\nu \bar{\theta} - \partial_\nu \theta \partial_\mu \bar{\theta}}{(1 + \bar{\theta}\theta)^2}, \tag{10}$$

so that the following duality condition must be fulfilled

$$^*F^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}, \tag{11}$$

which expresses the duality of $\mathcal{F}$ and $*\mathcal{F}$. The conditions for the existence of the pair ϕ, θ will be discussed later; for the moment let us say that they pose no difficulty. Equation (11) can be written also somewhat more formally as

$$*\phi^*\sigma = -\theta^*\sigma, \tag{12}$$

where $*$ is the Hodge or duality operator.

Note that, with the proper Lagrangian density, the field equations are the Maxwell's equations written in terms of the scalars ϕ and θ. They are nonlinear. It happens then that (6) and (9) (or equivalently (7) and (10)) play the role of the change of variables (1). For details seen reference [10].

4 Electromagnetic knots

In this section, we describe a curious kind of solution of Maxwell's equations in empty space. In order to do that, it is convenient to define *a pair of dual maps* $\phi, \theta : S^3 \mapsto S^2$, as two maps that verify equation (12). This means that the corresponding pull-backs of the area 2-form in S^2 define the two tensors $F_{\mu\nu}$ and ${}^*F_{\mu\nu}$, given by equations (7) and (10), which are dual in the sense of equation (11).

An important property of any pair of dual maps ϕ, θ is that the pull-backs $\mathcal{F} = -\sqrt{a}\,\phi^*\sigma$ and ${}^*\mathcal{F} = \sqrt{a}\,\theta^*\sigma$, verify necessarily the Maxwell's equations in empty space [10].

The maps $S^3 \mapsto S^2$ can be classified in homotopy classes, characterized by their Hopf index, a topological quantity that takes only integer values. It can be shown that the two Hopf indices are equal if tha maps are dual. It follows that the magnetic and the electric helicities are topologically quantized, since they are equal to

$$h_m = \int_{R^3} \mathbf{A}\cdot\mathbf{B}\, d^3r = na, \quad h_e = \int_{R^3} \mathbf{C}\cdot\mathbf{E}\, d^3r = n_e a, \tag{13}$$

where n is the Hopf index of both the maps ϕ, θ, and $\mathbf{A}$, $\mathbf{C}$ are the vector potentials for $\mathbf{B}$, $\mathbf{E}$, that is $\boldsymbol{\nabla}\times\mathbf{A} = \mathbf{B}$, $\boldsymbol{\nabla}\times\mathbf{C} = \mathbf{E}$.

Let us define now the concept of *electromagnetic knot*, as an electromagnetic knot in which any pair of magnetic lines (or of electric lines) is a pair of linked loops, except perhaps for some exceptional lines or times (see Figure 1).

From the mathematical point of view, *an electromagnetic knot* is an electromagnetic field generated by a pair of dual maps $\phi, \theta : S^3 \mapsto S^2$

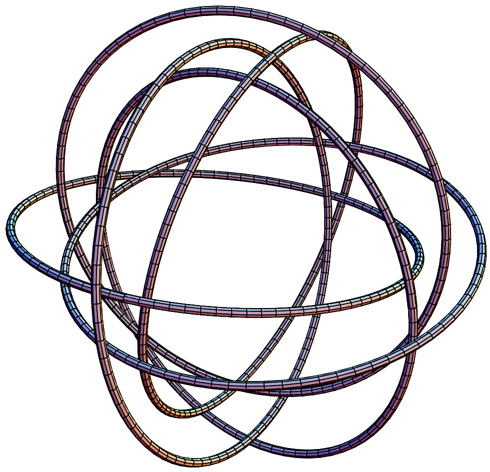

Figure 1: Schematic aspect of several force lines (either magnetic or electric) of an electromagnetic knot. Any two of the six lines shown are linked once.

verifying (11)-(12). (In other words, an electromagnetic field which can be expressed in terms of a pair of dual maps by means of equations (7) and (10)).

This definition implies that the corresponding Faraday 2-form $\mathcal{F}$ and its dual $*\mathcal{F}$ can be written as $\mathcal{F} = -\phi^*\sigma$ and $*\mathcal{F} = \theta^*\sigma$, i.e. as minus the pull-back and the pull-back of the area form in S^2 σ by the two maps. This will be relevant for the quantization of the charge.

A very important property is that the magnetic and electric lines of an electromagnetic knot are the level curves of the scalar fields $\phi(\mathbf{r}, t)$ and $\theta(\mathbf{r}, t)$, respectively. Another is that the magnetic and the electric helicities are topological constants of the motion, equal to the common Hopf index of the corresponding pair of dual maps ϕ, θ times a constant with dimensions of action times velocity.

In an electromagnetic knot, each line is labeled by a complex number. If there are m lines with the same label, we will say that m is the multiplicity. If all the pairs of line have the same linking number ℓ, it turns out that the Hopf index is gives as $n = \ell m^2$ [11, 12, 13, 14].

An electromagnetic knot is a radiation field (i.e. $\mathbf{E}\cdot\mathbf{B}=0$), the magnetic and electric lines being orthogonal at any point. This means that $\mathbf{E}$, $\mathbf{B}$ and the Poynting vector $\mathbf{S}=\mathbf{E}\times\mathbf{B}$, are three orthogonal vectors everywhere. The corresponding three families of curves (electric, magnetic and energy flux lines) form three orthogonal fibrations of S^3, since each family fills all the space, in the sense that a line of each kind passes through every point (although there may be some exceptional lines, or one or several of the vectors may vanish at certain points or times.) This property is allowed by the fact that S^3 is parallelizable. Indeed, an electromagnetic knot has a rich structure.

5 Two topological constants of the motion

A model of electromagnetism is possible in which all the radiation fields are electromagnetic knots as defined above. This model has been called *the topological model*. Surprisingly, it is equivalent to the standard Maxwell's theory in empty space or with point charges. A detailed presentation is given in rerefence [10]. It embodies, furthermore, two topological quantization laws [14]-[15].

5.1 Topological quantization of the electromagnetic helicity

The electromagnetic helicity is the semisum of the magnetic and electric helicities given by equations (13), $\mathcal{H}=(h_m+h_e)/2$. It is the Noether constant of the motion associated to the electromagnetic duality (the interchange of electricity and magnetism). We can express it as

$$\mathcal{H}=\int d^3k\ \left(\bar{a}_R(\mathbf{k})a_R(\mathbf{k})-\bar{a}_L(\mathbf{k})a_L(\mathbf{k})\right), \tag{14}$$

where $a_R(\mathbf{k}), a_L(\mathbf{k})$ are Fourier transform functions of the field $\mathbf{A}(\mathbf{r},t)$, which in QED are interpreted as the annihilation operators for right- and left-handed photons ($\bar{a}_R, \bar{a}_L$ being the corresponding creation operators). In the case of a knot, it follows that

$$n=\frac{1}{a}\int d^3k\ \left(\bar{a}_R(\mathbf{k})a_R(\mathbf{k})-\bar{a}_L(\mathbf{k})a_L(\mathbf{k})\right). \tag{15}$$

In QED, the integral in the right hand side of (14) and (15) is the operator for the difference between the numbers of right-handed and left-handed photons N_R-N_L. If the knots are classical, those Fourier transforms are functions, so that the integral in the right-hand side is the classical limit of this difference. Consequently, the value of $N_R - N_L$ for a knot is topologically quantized and takes the value *nac*. This suggests a criterion for the value of the normalizing constant: to take $a = 1$ in natural units (this is $a = \hbar c$ in the the rationalised MKS system and $a = \hbar c \epsilon_0$, in SU, ϵ_0 being the permittivity of empty space). One has then

$$n = N_R - N_L. \tag{16}$$

Equation (16) relates, in a very simple and appealing way, two meanings of the term helicity, relating to the wave and particle aspects of the field. At left, the wave helicity: the Hopf index n, characterizing the way in which the force lines — either magnetic or electric — curl around one another (as explained before $n = \ell m^2$, ℓ being the linking number and m the multiplicity of the map. At right, the particle helicity: the difference between the numbers of right-handed and left-handed photons. This is certainly a nice property. It suggests that the electromagnetic knots are worth of consideration. Note that this property gives a new interpretation of the number n. We knew that it is a Hopf index. We see that it is furthermore the difference of the classical limit of the numbers of right-handed and left-handed photons.

5.2 Topological quantization of the charge

The quantization of the electric charge is one of the most important and intriguing laws of physics. However, the value of the fundamental charge is obtained through experiments, all the efforts to predict it — or the fine structure constant α — within a theoretical scheme having failed so far.

This important law is usually stated by saying that the electric charge of any particle is an integer multiple of a fundamental value e, the electron charge, whose value in the International System of Units is $e = 1.6 \times 10^{-19}$ C. The Gauss theorem allows a different, although fully equivalent, statement of this property: the electric flux across any closed surface Σ which does not intersect any charge is always an integer multiple of e (we will use here the rationalized MKS system). This can be written as

$$\int_\Sigma \omega = ne, \tag{17}$$

where ω is the 2-form $\mathbf{E}\cdot\mathbf{n}\,dS$, $\mathbf{n}$ being a unit vector orthogonal to the surface, $\mathbf{E}$ the electric field and dS the surface element. We could as well write (17) as

$$\int_{\Sigma} *\mathcal{F} = ne, \tag{18}$$

$*\mathcal{F}$ being the dual to the Faraday 2-form $\mathcal{F} = \frac{1}{2}F_{\mu\nu}dx^{\mu}\wedge dx^{\nu}$. Stating in this way the discretization of the charge is interesting because it shows a close similarity with the expression of the topological degree of a map. Equation (9) shows that in the topological model with $a = 1$ ($a = \sqrt{\hbar c}$ in the MKS system)

$$\int_{\Sigma} *calF = \int_{\Sigma} \theta^{*}\sigma = n, \tag{19}$$

n being an integer number called the degree of the map $\Sigma \mapsto S^2$, induced by θ, which gives the number of times that S^2 is covered when one runs once through Σ (equal to the number of points in Σ in which θ takes any prescribed value).

The comparison of (18) and (19) shows that there is a close formal similarity between the dual to the Faraday 2-form and the pull-back of the area 2-form of a sphere S^2. It can be expressed in this way. Let an electromagnetic field be given, such that its form $*\mathcal{F}$ is regular except at the positions of some point charges. Let a map $\theta : R^3 \mapsto S^2$ be also given, which is regular except at some point singularities where its level curves converge or diverge. It happens then that equations (18) and (19) are simultaneously satisfied for all the closed surfaces Σ which do not intersect any charge or singularity.

This means that the electric charge is topologically quantized inthe topological model because of the consideration of $*\mathcal{F}$ as an area 2-form, implicit in (9). Furthermore, the fundamental charge is equal to $\sqrt{a}$, and the number of fundamental charges in a volume has the meaning of a topological index.

It is easy to understand that $n = 0$ if θ is regular in the interior of Σ. This is because each level curve of θ (*i. e.* each electric line) is labeled by its value along it — a complex number — and, in the regular case, any one of these lines enters into this interior as many times as it goes out of it. But assume that θ has a singularity at point P, from which the electric lines diverge or to which they converge. If Σ is a sphere around P, we can identify R^3 except P with $\Sigma \times R$, so that the induced map $\theta : \Sigma \mapsto S^2$ is regular. In this case, n need not vanish and is equal to the number of times that θ takes any prescribed complex value in Σ, with due account to the orientation. Otherwise stated, among the electric lines diverging from or converging to P, there are $|n|$ whose label is equal to any prescribed complex number.

This shows why the topological model embodies a topological quantization of the charge. This mechanism for the quantization of the charge was first shown in 1991 and developed later in [15].

To understand better this mechanism of discretization, let us take the case of a Coulomb potential [15, 14], $\mathbf{E} = Q\mathbf{r}/(4\pi r^3)$, $\mathbf{B} = 0$. The corresponding scalar is then

$$\theta = \tan\left(\frac{\vartheta}{2}\right) \exp\left(i\frac{Q}{\sqrt{a}}\varphi\right), \tag{20}$$

where φ and ϑ are the azimuth and the polar angle. The scalar (20) is well defined only if $Q = n\sqrt{a}$, n being an integer. The lines diverging from the charge are labeled by the corresponding value of θ, so that there are $|n|$ lines going in or out of the singularity and having any prescribed complex number as their label. If $n = 1$, it turns out that $\theta = (x + iy)/(z + r)$.

This mechanism has a very curious aspect: it does not apply to the source but to the electromagnetic field itself. This is surprising; one would expect that the topology should operate restricting the fields of the charged particles. However, in this model, it is the field who mediates the force the one which is submitted to a topological condition. It must be emphasized furthermore that the maps $S^3 \mapsto S^2$, given by the two scalars ϕ, θ are regular except for singularities at the position of point charges, either electrical or magnetic (if the latter do exist). At these points, the level curves (*i. e.* the electric lines) converge or diverge.

We see that the topological model predicts that the fundamental charge has the value $q_0 = 1$ (in natural units) or

$$q_0 = \sqrt{\hbar c}, \tag{21}$$

(in the MKS system) which is about 3.3 times the electron charge. In the ISU, this is $q_0 = \sqrt{\hbar c \epsilon_0} = 5.29 \times 10^{-19}$ C. Note that the same discretization mechanism would apply to the hypothetical magnetic charges (located at singularities of ϕ), their fundamental value being also $q_0 = \sqrt{\hbar c}$.

6 Hidden nonlinearity

We have found a structure with two levels. At the deeper one, it is nonlinear since the scalars ϕ and θ obey highly nonlinear equations. However the transformation $T : \sigma \to \mathcal{F}, *\mathcal{F}$ given by (9) and (12)

$$T : \sigma \to (\mathcal{F} = -a\phi^*\sigma,\ *\mathcal{F} = a\theta^*\sigma)\,, \tag{22}$$

σ being the area two-form in S^2, changes these nonlinear equation for ϕ and θ into the linear Maxwell's ones for $\mathcal{F}$, linearizing thus the theory. This is important: the Maxwell's equations are the exact linearization of a nonlinear and topological theory (by change of variables, not by truncation!). The theory seems to be linear if one looks to the equation satisfied by the field $F_{\mu\nu}$, but it cannot be really linear since the topological quantization of the helicity imposes the nonlinear conditions

$$h_m = \int \mathbf{A}\cdot\mathbf{B}\, d^3r = na, \quad h_e = \int \mathbf{C}\cdot\mathbf{E}\, d^3r = na. \tag{23}$$

It is clear therefore that one cannot obtain another solution just by multiplying $\mathbf{B}$ and $\mathbf{E}$ by a real number (or by adding two different solutions.)

This property was termed *hidden nonlinearity* in previous work. It is due to the fact that the transformation T is not invertible, since there are solutions of Maxwell's equations for which $T^{-1}\mathcal{F}$ is not defined. In other words, in some cases there are no scalar fields ϕ, θ generating $\mathcal{F}$, which could be interpreted as maps $S^3 \mapsto S^2$ (as the examples given before show clearly). As a consequence, although all the electromagnetic fields of the topological model obey the linear Maxwell's equations, *they do not span the vector space of all the solutions, but form a nonlinear subset instead.*

Which are the standard electromagnetic fields which must be excluded from the topological theory because they cannot be generated by a pair of dual maps? First of all, those with helicities not verifying the equations (23); also those for which the scalars do exist locally but do not behave well at infinity or are not of class C^1 and the Hopf index can not be defined. Contrary to what it might seem, this is not necessarily a drawback of the model. In fact, it can be said the Maxwell's equations have too many solutions, since not all of them can be realized in nature. Some because their energy, or their momentum, is infinite. Others are Coulomb or Liener-Wiechert potentials coupled to charges which are not integer multiples of the electron fundamental value e, or they would have been radiated by monopoles (if these particles do not exist), or have discontinuities in surfaces, meant to represent in a simple way changes of the field which are abrupt but continuous. Consequently, that not all the standard solutions are included in the topological model is not necessarily a bad thing.

7 Conclusion

There is a topological structure below Maxwell's equation in empty space or coupled to point charges. A consequence is that they can be obtained as the linearisation by change of variables of a nonlinear theory based on two scalars that are constant, along the magnetic and electric force lines, respectively. In the topological model, the theory is still linear, but there is some subtle form of nonlinearity, thanks to which the linearity is compatible with the existence of the topological constants of the motion equal to the electromagnetic helicity and the electric (and eventually the magnetic) charge.

References

[1] E. Whittaker, *A History of the Theories of Aether and Electricity* (1910) (reprinted by Humanities Press Inc, New York, 1973).

[2] Lord Kelvin (then W. Thomson), Trans. R. Soc. Edin. **25**, 217-260 (1868)

[3] Th. Archibald, Mathematical magazine **62**, 219 (1989)

[4] M. Atiyah, *Geometry, topology and physics, Q. Jl. R. Astr. Soc.* **29**, 287-299 (1988)

[5] J.C. Maxwell, "Atomism", *Enciclopaedia Britannica*, II, 9th edition, 1875; also in J. C. Maxwell, *Scientific papers*, edited by J. M. Sánchez Ron (CSIC, Madrid, 1998)

[6] J.Z. Buchwald, "William Thomson (Lord Kelvin)", in *Dictionary of Scientific Biography*, Ch. Coulson (ed.), vol. 13-14, (Ch. Scribner's Sons, New York, 1980)

[7] Lord Kelvin (then W. Thomson), "Lecture to the Royal Society of Edinburgh", 18 February 1967, Proc. R. Soc. Edinb. 6, 94 (1869); see also S. P. Thomson, *Life of Lord Kelvin* (Macmillan, London, 1910) p 517

[8] M. Atiyah, *The geometry and physics of knots* (Cambridge University Press, 1990)

[9] H. K. Moffatt, Nature **347** 321 (1990)

[10] A. F. Rañada and J. L. Trueba, "Topological Electromagnetism", to appear in the book *Contemporary Optics and Electrodynamics*, *Advances in Chemical Physics*, vol 114, M. Evans, ed., John Wiley & Sons, New York.

[11] A. F. Rañada, Lett. Math. Phys. **18**, 97-106 (1989)

[12] A. F. Rañada, J. Phys. A:Math Gen **25**, 1621-1641 (1992)
[13] A. F. Rañada, J. Phys. A:Math. Gen. **23**, L815-820 (1990)
[14] A.F. Rañada and J.L. Trueba, Phys. Lett. A 202, 337-342 (1995); Phys. Lett. A 235, 25-33 (1997)
[15] A. F. Rañada, An. Fis. (Madrid) A 87, 55-59 (1991); A.F. Rañada and J.L. Trueba, Phys. Lett. B 422, 196-200 (1998); A. F. Rañada, http://xxx.lanl.gov/list/hep-th/9904158 (1999)

(Manuscrit reçu le 29 novembre 2000)