Alexey Stakhov, Anna Sluchenkova

Museum of Harmony and the Golden Section

MATHEMATICAL CONNECTIONS IN NATURE, SCIENCE, AND ART

ABSTRACT

В статье рассматривается концепция Музея Гармонии и Золотого Сечения как уникального естественнонаучного, исторического и художественного Музея, представляющего собой коллекцию всех созданий Природы, Науки и Искусства, основанных на Золотом Сечении http://www.goldenmuseum.com/

In the article we consider a concept of the Museum of Harmony and the Golden Section as unique history-nature-science-art museum, collection of the Nature, Science and Art works based on the Golden Section http://www.goldenmuseum.com/

1. A CONCEPT OF HARMONY AND THE GOLDEN SECTION

Throughout the history people aspire to surround themselves with beautiful things. At some point the question arose: What is the basis of a Beauty? Ancient Greeks developed the science of *aesthetics* as a way to analyze a Beauty, believing that a Harmony is its basis. Beauty and Truth are interrelated: an artist searches for the Truth in the Beauty, and a scientist for the Beauty in the Truth.

Is it possible to compare the beauty of a sculpture, a temple, a picture, a symphony, a poem, or a nocturne? If a formula could be found, then the loveliness of a chamomile flower and a naked body could be measured and compared. The well-known Italian architect Leone Battista Alberti spoke about the Harmony as follows: *"There is something greater, composed of combination and connection of three things (number, limitation and arrangement), something that lights up the face of beauty. And we called it Harmony, which is, doubtlessly, the source of some charm and beauty. You see assignment and purpose of Harmony in arranging the parts, generally speaking, different in their nature, by certain perfect ratio so that they meet one another creating beauty ... It encompasses all human life, penetrates through the nature of things. Therefore everything that is made by Nature is measured by the law of Harmony. Also there is no greater care for the Nature than that of everything created by it to be perfect. It is impossible to achieve this without Harmony; therefore without it the greatest consent of the parts is disintegrated".*

There are many well-known "formulas of beauty" such as certain geometrical shapes: square, circle, isosceles triangle, and pyramid. However, the most wide-spread criterion of beauty is one unique mathematical proportion called the Divine Proportion, Golden Section, Golden Number, or Golden Mean. The Golden Section and related to it Fibonacci numbers (1, 1, 2, 3, 5, 8, 13, …) permeate the history of art. Examples of well known works, which exhibit this proportion, are Khufu's Pyramid of Egypt, the Parthenon in Athens, Greek sculpture, the "Mona Lisa" by Leonardo da Vinci, paintings by Rafael, Shishkin, and the modern Russian artist Konstantin Vasiljev, Chopin's etudes, music of Beethoven and Mozart, "Modulor" by Corbusier.

The Museum of Harmony and the Golden Section [1] contains a vast collection of information on the Golden Section in nature, science, and art. In virtual form, the Museum can be seen on the Web at http://www.goldenmuseum.com/. The main goal of the Museum is given in the introduction: *"The 'Golden Proportion' is a mathematical concept and its analysis is first of all a problem of science. But it is a criterion of Harmony and Beauty, and this is already* *category of Art and Aesthetics. And our Museum, which is dedicated to analysis of this unique phenomenon, is doubtlessly, a scientific museum dedicated to the analysis of harmony and beauty from the mathematical point of view."*

The Museum includes two main parts: *cognitive* and *scientific*. The former part aims to acquaint all people—students, teachers, engineers, specialists in various areas of science, artists, musicians, and representatives of all arts—with surprising discoveries of ancient science: the Golden Section and its various applications. The scientific part of the Museum aims to give information on modern scientific discoveries based on the Golden Section.

The Museum consists of the following halls:

(1) The Golden Section in History of Culture
(2) The Golden Section, Nature and Man
(3) The Golden Section in Art
(4) Mathematics of Harmony
(5) Fibonacci Computers
(6) Fibonaccization of Modern Science
(7) Harmonic Education

2. THE GOLDEN SECTION IN THE HISTORY OF CULTURE

This hall of the Museum consists of the following exhibitions:

(1) What is the meaning of the Golden Section and Fibonacci Numbers?
(2) The Golden Section in history of Ancient Art
(3) Fibonacci numbers and the Golden Section in the Middle Ages and Renaissance
(4) The problem of Harmony and Symmetry in the 19^{th} century science.

Although the material of every exhibition is well known separately, the collection of facts concerning the golden section confirms the outstanding role it plays in the history of culture. Let us consider the basic exhibitions of the Museum, which carry scientific evidence of the role of the golden section in the history of material and spiritual culture.

2.1. The Golden Section. Johannes Kepler said that geometry has two treasures: one of them is Pythagorean Theorem, the other one is the golden section. The former can be compared to a measure of gold, the latter to a precious jewel.

The golden section arises from the division of the line-segment *AB* by the point *C* in the extreme and mean ratio (Fig.1) that is,

$$\frac{AB}{CB} = \frac{CB}{AC} \quad (1)$$

Figure 1. The Golden Section

It is reduced to the equation:

$$x^2 = x + 1 \quad (2)$$

The positive root of the equation

$$\tau = \frac{1+\sqrt{5}}{2} \approx 1{,}618$$

is called the *golden ratio* and the division of the line-segment in the ratio of (1) is called the *golden section*.

Being the root of the equation of (2), the golden ratio has the following wonderful property:

$$\tau^2 = \tau + 1 \tag{3}$$

The expression of (3) can be rewritten as

$$\tau = 1 + \frac{1}{\tau} \tag{4}$$

$$\tau - 1 = \frac{1}{\tau} \tag{5}$$

Hence, by subtracting of 1 from $\tau = 1{,}618$ we get the reciprocal to the golden ratio

$$\frac{1}{\tau} = 0{,}618\,.$$

It was proved that the golden ratio is the only positive number having this property.

It should be noted that the numbers of 1,618 and 0,618 are supposed to express a proportion of the golden section or the golden ratio.

Representation of the Golden Ratio in the form of "continued" fraction

Let us prove now one more surprising property of the golden ratio, which results from the identity (4). If in the right-hand part of (4) we substitute τ by its value given (4), we will come to representation of τ in the form of the following "multistoried" fraction:

$$\tau = 1 + \frac{1}{1+\frac{1}{\tau}}\,. \tag{6}$$

If we continue such substitution many times in the right-hand part of (6) we will get the following "multistoried" fraction with infinite number of "stories":

$$\tau = 1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ldots}}}} \tag{7}$$

The representation of (7) is called in mathematics "continued" or "chain" fraction. Note that the theory of "continued" fractions is one of the significant parts of modern mathematics.

Representation of the Golden Ratio in "radicals"

Let us consider now once again the identity of (3). It can be represented in the following form:

$$\tau = \sqrt{1+\tau} \tag{8}$$

If in the right-hand part of the identity (8) we substitute now τ by the same expression of (8), we will get the following representation for τ:

$$\tau = \sqrt{1 + \sqrt{1 + \tau}} \qquad (9)$$

If we substitute again τ in the right-hand part of the identity (9) by the same expression of (8) and repeatedly, we will get one more remarkable representation of the golden ratio in "radicals":

$$\tau = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \ldots}}}} \,. \qquad (10)$$

Every mathematician intuitively aims to express mathematical results in the simplest, compact form. And if he finds such form, he enjoys "aesthetic pleasure". In this respect (in tendency to "aesthetic" expression of mathematical outcomes) the mathematical creativity is similar to creativity of composer or poet, whose main problem consists of obtaining perfect musical or poetic forms, which would give us "aesthetic pleasure". Note, that the formulas (7) and (10) produce also "aesthetic enjoying" and invoke feeling of rhythm and harmony, when we begin to think about infinite repeatability of the same simple mathematical elements in the formulas for τ given by (7) and (10).

Pentagon and pentagram

The golden section is widely used in geometry. It is proved that

$$\tau = \frac{1 + \sqrt{5}}{2} = 2 \cos 36°.$$

Using this formula we can show that in the regular pentagon *ABCDE* the cross points of the diagonals *F, G, H, K, L* divide them in the golden section and form the new pentagon *FGHKL* (Fig.2).

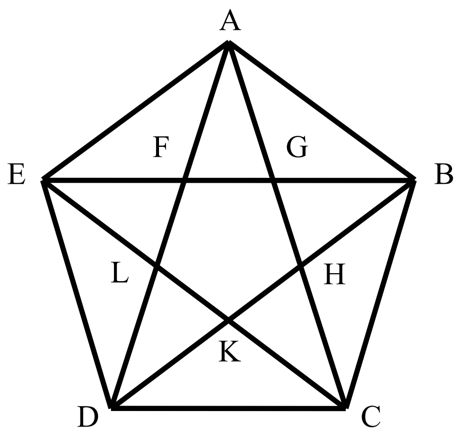


Figure 2. Regular pentagon.

The "golden" cup and the "golden" triangle

The regular pentagon comprises a number of wonderful figures, which are widely used in works of art. In ancient Egypt and classic Greece the law of the "golden cup" was well known. It was used by architects and goldsmiths. If we draw the diagonals *BE, BD* and *EC* in the pentagon *ABCDE* (Fig.3), the dashed part receives a form of the "golden cup", which can be expressed by means of the following ratios:

$$\frac{EF}{FC} = \frac{EC}{EF} = \frac{EB}{DC} = \tau .$$

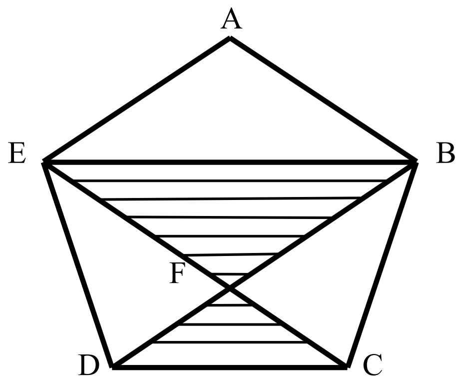


Figure 3. The "golden" cup.

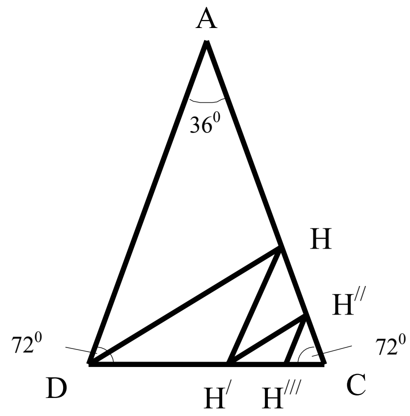


Figure 4. The golden triangle.

There is another graceful figure enclosed into the pentagon. It is the golden triangle, for example, *ADC* (Fig.4), whose base is the side of the regular pentagon. The triangle has the vertex angle measuring 36° and the base angles measuring 72° each. The Pythagoreans were greatly excited by the fact that the bisector *DH* of the angle *D* coincides with the diagonal *DB* of the pentagon and the point *H* divides the side *AC* in the golden section. So, the new smaller golden triangle *DCH* appears. If we draw the bisector of the angle *H* to the point *H′* on the side *DC*, then the bisector of the angle *H′* to the point *H″* on the side *AC* and continue this procedure endlessly, we get an infinite sequence of the golden triangles.

The "golden" rectangle

The same property is inherent in the golden rectangle *ABCD* (Fig.5), which ratio of the sides *AB : AD* equals to the golden ratio.

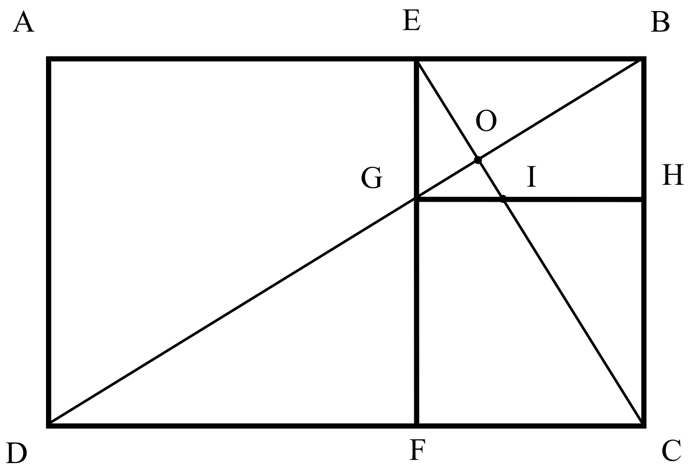


Figure 5. The golden rectangle.

Deriving the square *AEFD* from the rectangle *ABCD* we get the new golden rectangle

EBCF, which ratio of the sides *EF:EB* equals to the golden ratio. If we continue the procedure endlessly we get an infinite sequence of the squares and the golden rectangles.

The "Golden" spiral

The spiral is a plane line derived by a driving point, which moves away according to a definite law from the beginning of the ray and uniformly rotates around the beginning. If we assume the beginning of the spiral as the pole of the polar coordinate system then mathematically the spiral can be presented with the help of some polar equation $\rho = f(\varphi)$, where ρ is the radius-vector of the spiral, φ is the angle put aside on the polar axis, $f(\varphi)$ is some monotonically increasing or decreasing positive function. If the point moves away from the beginning uniformly ($\rho = a\varphi$) we have *Archimedes spiral*. If the point moves away according to the *exponential law* ($\rho = ae^{m\varphi}$ where a is an arbitrary positive number we have an *equiangular spiral* (Fig.6).

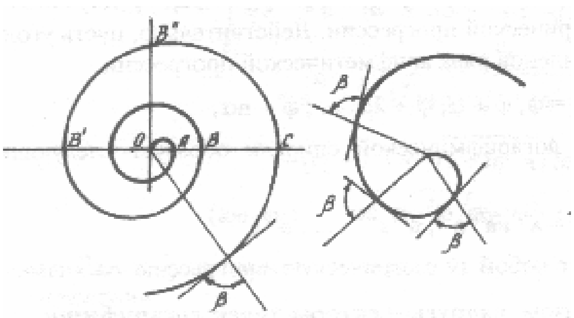


Figure 6. Equiangular spiral

The equiangular spiral has a number of interesting properties:

1. In the equiangular spiral the line segments *OA*, *OB*, *OC*, *OD*, ... derives geometrical progression, that is

$$\frac{OA}{OB} = \frac{OB}{OC} = \frac{OC}{OD} = \dots = m,$$

where m is a denominator of the progression.

2. The radius-vector and tangent to any point of the equiangular spiral form a constant angle β, that is, the curve intersects all rays coming out of the pole *O* under the same angle.
3. The equiangular spiral is degenerated accordingly to a straight line and circumference with values of the angles $\beta = 0$ and $\beta = 90°$. This means that the spiral has properties of both straight line and circumference.

Any equiangular spiral represents the scheme of growth or ascending and can be expressed by geometrical progression. Here the "golden" equiangular spiral is of a special interest. In this spiral the terms of geometrical progression corresponding to the spiral are the degrees of the golden proportion $\{\tau^n\}(n = 0, \pm1, \pm2, \pm3, ...)$. Such spiral has a property to be simultaneously geometric and arithmetic progression. This means that its exponential growth is provided by simple addition of the two adjacent terms. In opinion of many researchers, this remarkable property (a possibility of implementation of ascending by simple addition) enables to explain many phenomena and processes in botanic and biology. Note also, that the "golden" spiral is inscribed into the "golden" rectangle in natural mode (Fig.7).

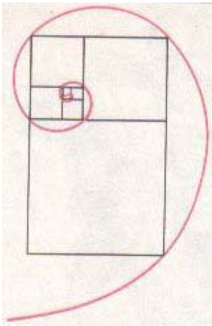


Figure 7. The "golden" spiral

If as the beginning of the spiral we select the point, to which "golden" rectangles sequentially converge, the "golden" spiral will pass through three of four tops of each "golden" rectangles sequentially constructed on Fig. 7.

Dodecahedron and icosahedron

The golden section is closely connected to the so-called *Platonic solids*, in particular to *dodecahedron* and *icosahedron* (Fig.8). The dodecahedron has 12 faces, 30 edges and 20 vertices. Each face of the dodecahedron is the regular pentagon and has 5 plane angles. Thus, a total number of plane angles of the dodecahedron surface equals to the number of 60=5×12. On the other hand, 60=3×20. The latter means that three neighboring plane angles converge at each dodecahedron vertex. Finally, the face number of 12 multiplied by the edge number of 30 equals to the number of 360.

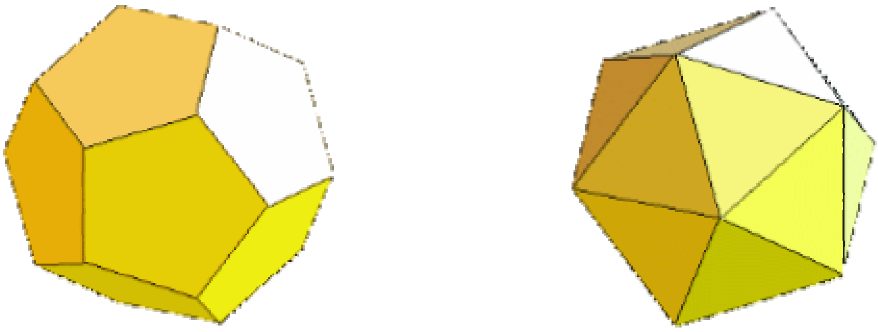


Figure 8. Dodecahedron and Icosahedron

If we look at the dodecahedron we can see that the 12 corners become the 12 centers of each of the 12 pentagons that form the faces of the dodecahedron. If we look at the icosahedron we can see that the 12 corners can also become the 12 points of each of the 20 triangles that form the faces of the icosahedron. As for the icosahedron, it is the Platonic solid, which is "polar" to the dodecahedron. Both the icosahedron and the dodecahedron have the same edge number of 30. Besides, the icosahedron face number of 20 equals to the vertex number of the dodecahedron and the vertex number of 12 of the icosahedron equals to the face number of the dodecahedron. As five plane angles converge in each icosahedron vertex, a total number of plane angles equals to the number of $60=5\times12$ and the product of the edge number by the vertex number equals to 360. The icosahedron also has a relation to a regular pentagon and, therefore, to the golden ratio because the outer edges of the five neighboring triangles, which converge in any icosahedron vertex, make the regular pentagon.

2.2. Phenomenon of Ancient Egypt. Early in the 20th century in Saqqara (Egypt), archaeologists opened the crypt, in which the Egyptian architect Khesi-Ra (Khesira) was buried.

The wood boards-panels covered by a magnificent thread were extracted from the crypt alongside with different material assets. In total there were 11 boards in the crypt; among them only 5 boards were preserved; the remaining panels were completely destroyed by moisture in the crypt.

All preserved panels depict the architect Khesi-Ra who is surrounded by different figures having symbolical significance (Fig.9). For long time the assignment of Khesi-Ra's panels was vague. At first the Egyptologists considered these panels as false doors. However, since the 60th years of the 20th century situation with the panels began to be elucidated. In the beginning of the 60th the Russian architect Shevelev paid his attention to the fact that the staffs the architect holds in his hands on one of the panels relate between themselves as $1:\sqrt{5}$, that is, as the ratio of the small side and the diagonal of the rectangle with the side ratio of 1:2 ("two-adjacent square"). Just this observation became the initial point for research of the other Russian architect Shmelev who made careful geometrical analysis of Khesi-Ra's panels and as a result came to the sensational discovery described in the brochure *"Phenomenon of Ancient Egypt"* (1993) [2].

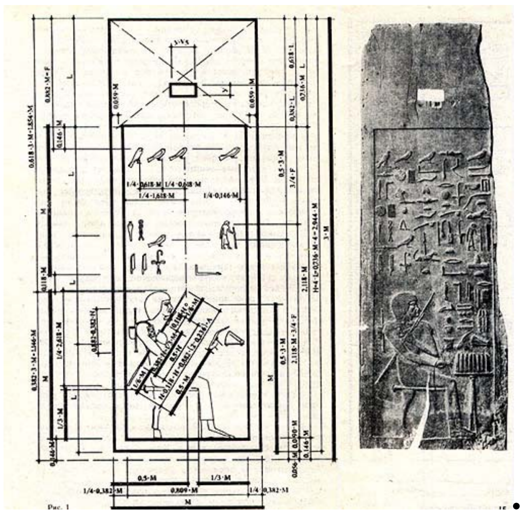


Figure 9. Khesi-Ra's panel

After exploring Khesi-Ra's panels Igor Shmelev made the following discovery: *"But now, after the comprehensive and argued analysis by the method of proportions we get good causes to assert that Khesi-Ra's panels are the harmony rules encoded in geometry language….*

So, in our hands we have the concrete material evidence, which shows us by "plain text" the highest level of abstract thinking of the Ancient Egypt intellectuals. The artist who cut the panels with amazing accuracy, jeweler refinement and masterly ingenuity demonstrated the rule of the 'Golden Section' in its broadest range of variations. In outcome it was born the 'GOLDEN SYMPHONY' presented by the ensemble of the highly artistic works, which testifies not only ingenious talents of their creator, but also convincingly verifies that the author was let to the secret of harmony. This genius was of the 'Golden Business Craftsman' by the name of Khesi-Ra.".

But who was Khesi-Ra? The ancient texts inform us that Khesi-Ra was *"a Chief of Destius and a Chief of Boot, a Chief of doctors, a writer of the pharaoh, a priest of Gor, a main architect of the pharaoh, a Supreme Chief of South Tens, and a carver".*

Analyzing the above listed Khesi-Ra's regalia Shmelev pays special attention to the fact that Khesi-Ra was the priest of Gor. In the Ancient Egypt Gor was considered as the God of Harmony and therefore to be the priest of Gor meant to execute functions of the keeper of Harmony.

As follows from his name, Khesi-Ra had been elevated to the rank of the God of Ra (God of the Sun). Shmelev suggests that Khesi_Ra could get this high award for *"development of aesthetic ... principles in the canon system, which reflects the harmonic fundamentals of the Universe ... The orientation on the harmonic principle discovered by the Ancient Egypt civilization was the path to unprecedented flowering of culture; this flowering falls into period of Zoser-pharaoh when the system of written signs was completely implemented. Therefore it is possible to assume that Zoser's pyramid became the first experimental pyramid, which was followed by construction of the unified complex of the Great pyramids in Giza according to the program designed under Khesi-Ra supervision".*

And let us consider one more quotation from Shmelev's brochure [2]: "*It is only necessary to recognize that the Ancient Egypt civilization is the super-civilization explored by us extremely superficially and it demands a qualitatively new approach to development of its richest heritage.... The outcomes of researches of Khesi-Ra's panels demonstrate that the sources of modern science and culture are in boundless stratums of a history feeding creativity of the craftsmen of our days with great ideas, which for long time inspired aspirations of the outstanding representatives of the mankind. And our purpose is not to lose a unity of a binding thread."*

2.3. Mysteries of the Egyptian Pyramids. Iinfinite, uniform sea of sand, infrequent dried bushes of plants, hardly noticeable tracks from an elapsed camel are swept with a wind. The incandescent sun of wasteland … And it seems dull, as if covered with fine sand.

And suddenly, as a mirage, before the amazed look there arise pyramids (Fig.10), fancy rock figures directed toward the Sun. By their vast sizes, perfection of the geometric form they strike our imagination. According to many descriptions, these gigantic monoliths earlier had different view than presently. They shined on the Sun with white glaze of the polished calcareous tables on the background of many-pillar adjacent temples. Near the pharaohs' pyramids there were the pyramids of pharaohs' wives and other members of their families. Pharaoh's authority in Ancient Egypt was huge, divine honors were given to him, the pharaoh was called the "Great God". The God-Pharaoh was a Promoter of country, a Judge of people fates. Cult of the died pharaoh gained huge importance in the Egyptian religion. The gigantic pyramids were constructed for preservation of pharaoh's body and his spirit and for extolling his authority. And it is not without reason that these works of human hands fall into one of the seven miracles of the World.

Figure 10. Complex of pyramids in Giza

The assignment of pyramids was multifunctional. They are served not only burial vaults of pharaohs, but also were attributes of majesty, power and riches of country, monuments of culture, storehouses of the country history and items of information on life of pharaohs and people.

It is clear, that the pyramids had deep "scientific contents" embodied in their forms, sizes and orientation on terrain. Each part of a pyramid, each element of the form were selected carefully and had to demonstrate high level of knowledge of the creators of pyramids. They were constructed to last millennia, "for all time". And it is not without reason the Arabian proverb claims: *"All in the World is afraid of the Time. The Time is afraid of the Pyramids ".*

Among the gigantic Egyptian pyramids the Great Pyramid of the pharaoh Khufu is of special interest. Before analyzing the shape and sizes of Khufu's pyramid it is necessary to remind ourselves of the Egyptian measure system. The Ancient Egyptians used three measure units: "elbow" (466 mm) equaling to 7 "palms" (66,5 mm), which, in turn, was equal to 4 "fingers" (16,6 mm).

Let us regard geometric analysis of the sizes of Khufu's pyramid (Fig.11) following the reason given in the remarkable book of the Ukrainian scientist Nickolai Vasutinski "The Golden Proportion" (1990) [3].

The majority of researchers believe that the length of the side of the pyramid basis for example, GF is equal to L = 233,16 m. This value corresponds almost precisely to 500 "elbows". We would lave full correspondence to 500 "elbows" if we considered the length of "elbow" equal to 0,4663 m. The altitude of the pyramid (H) is estimated by researchers variously from 146,6 m up to 148,2 m. And depending on an adopted pyramid altitude all the ratios of its geometric elements change considerably. What is a cause of distinctions in estimation of the pyramid altitude? Strictly speaking, Khufu's pyramid is truncated. Its topic platform today has the size approximately 10×10 m, but one century back it was equal to 6×6m. Apparently, that the top of the pyramid was dismantled and it does not correspond to the initial pyramid.

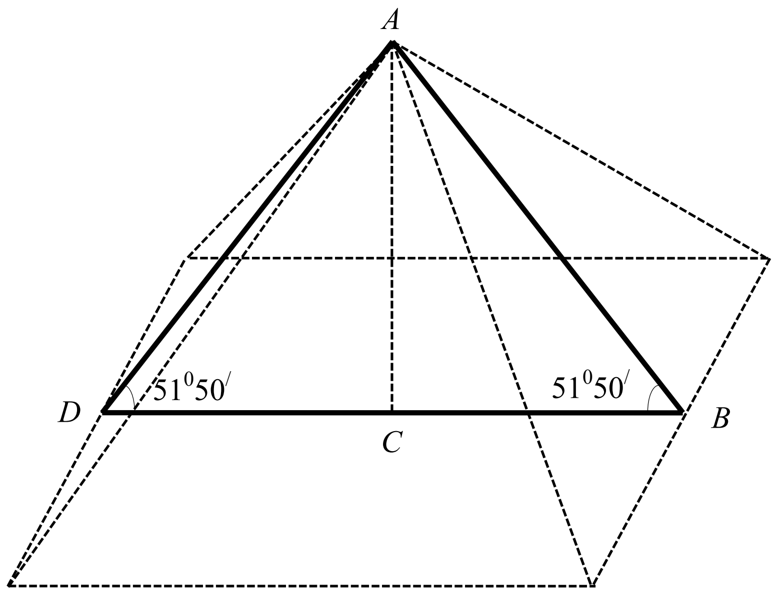


Figure 11. Geometric model of Khufu's pyramid

Estimating the pyramid altitude it is necessary to take into consideration such physical factor as "shrinkage" of construction. For a long time under effect of enormous pressure (reaching 500 tons for 1 м^2 of undersurface) the pyramid altitude decreased in comparison to its initial altitude.

What was the initial altitude of the pyramid? This altitude can be reconstructed if we find the main "geometrical idea" of the pyramid.

In 1837 the English colonel G. Vaise measured the inclination angle of the pyramid faces: it appeared equal to α=51°51′. The majority of researchers recognize this value up to now. The indicated value of the inclination angle corresponds to the tangent equal to 1, 27306. This value corresponds to the ratio of the pyramid altitude *AC* to the half of its basis *CB* (Fig.11), that is, $AC/CB = H/(L/2) = 2H/L$.

And here the researchers met large surprise! If we take a square root of the "golden" proportion $\sqrt{\tau}$ we get the following outcome $\sqrt{\tau}$ = 1,272. Comparing this value with value tgα = 1,27306 we can see that these values are very close. If we take the angle α=51°50′, that is, decrease it by one arc minute the value of tgα will become equal to 1,272, that is, will be equal to the value of $\sqrt{\tau}$. It is necessary to note, that in 1840. G. Vaise repeated his measurements and corrected the value of the angle to α=51°50′.

These measurements resulted in the following rather interesting hypothesis: the ratio $AC/CB = \sqrt{\tau}$ = 1,272 was put in the basis of the triangle *ACB* of Khufu's pyramid! If we mark now the lengths of the triangle *ABC* sides through *x, y, z*, and also take into consideration that the ratio $y/x = \sqrt{\tau}$ then according to the Pythagorean Theorem the length *z* can be computed as the following:

$$z=\sqrt{x^2+y^2}\ .$$

If we take x=1, y=$\sqrt{\tau}$, then

$$z=\sqrt{1+\tau}=\sqrt{\tau^2}=\tau$$

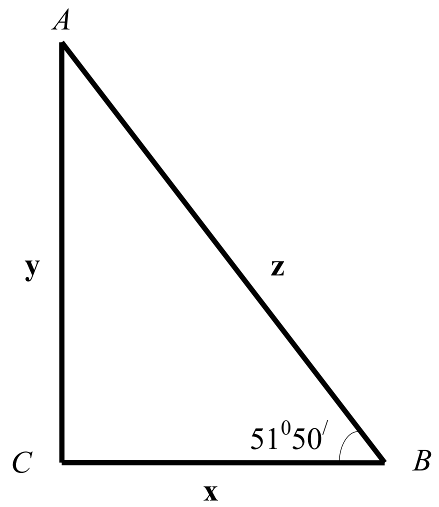


Figure 12. The “golden” right triangle

The right triangle, which ratios of the sides are τ: $\sqrt{\tau}$:1, is called the "golden" right triangle (Fig.12). Then if we accept a hypothesis that the "golden" right triangle is the main “geometrical idea" of Khufu’s pyramid it is possible to compute the "design" altitude of Khufu’s pyramid. It is equal to $H = (L/2)\times\sqrt{\tau}$ = 148,28 м.

Let us deduce now some other relations for Khufu’s pyramid, which result from the "golden" hypothesis. In particular, let us find the ratio of the external area of the pyramid to the area of its basis. For this purpose we take the length of the leg *CB* as the unit, that is, *CB* = 1. But then the length of the side of the pyramid basis *GF* = 2 and the area of the basis *EFGH* will be equal to S_{EFGH} = 4. Let us calculate now the area of the lateral face of Khufu’s pyramid. As the altitude of *AB* of the triangle *AEF* is equal to τ then the area of each lateral face will be equal to S_Δ = τ. Then the common area of all the four lateral faces of the pyramid will be equal to 4τ and the ratio of the common external area of the pyramid to the area of its basis will be equal to the “golden” proportion! This also is the main geometrical secret of Khufu’s pyramid!

The analysis of the other Egyptian pyramids demonstrates that the Egyptians always aimed to embody some relevant mathematical knowledge in pyramids. In this respect Khafre’s pyramid is rather interesting. The measurements of Khafre’s pyramid showed that the inclination angle of the lateral faces is equal to 53°12′ that corresponds to the leg ratio of the right triangle: 4:3. Such leg ratio corresponds to the well-known right triangle with the side ratios: 3:4:5; this one is called "perfect", "sacred" or "Egyptian" triangle. According to historians testimony the "Egyptian" triangle had magic meaning. Plutarch wrote that the Egyptians compared nature of the Universe to the "sacred" triangle; they symbolically assimilated the vertical leg to a husband, the basis to a wife, and the hypotenuse to a child who is born from both.

According to the Pythagorean Theorem we have: $3^2 + 4^2 = 5^2$ for the right triangle with side ratios: 3:4:5. It is possibly that the Egyptian priests wanted to perpetuate just this theorem in carrying up the pyramid based on the right triangle 3:4:5? It is difficult to find a more successful example for demonstration of the Pythagorean Theorem, which was well known for Egyptians long before its rediscovery by Pythagoras. Thus, the ingenious designers of the Egyptian pyramids aimed to strike their far offsets by depth of their mathematical knowledge, and they have reached this by selecting the "golden" right triangle as the main "geometrical idea" of Khufu's pyramid and of the "sacred" or "Egyptian" triangle as the main "geometrical idea" of Khafre's pyramid.

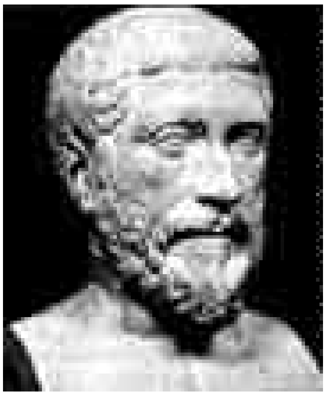
2.4. Mystery of the Egyptian Calendar. The Egyptian calendar, created in the 4th millennium B.C., was one of the first solar calendars. One year of the Egyptian calendar had the following structure: 365 = 12×30+5. It meant that the Egyptian calendar consisted of 12 months by 30 days in each month plus 5 holidays, which were added to current year consisted of 360 days and which did not entered to any month.

Several questions arise about the principal idea of this calendar. Why did the Egyptians choose 12 months in the year? Why did every month have exactly 30 days? Other calendars, such as the Mayan, consisted, for example, of 18 months by 20 days in each. Similar questions concern the Egyptian system of time and angle measurement, in which the numbers 12, 30, 60, 360 recur again. Why was a circle circumference divided into 360 degrees ($2\pi = 360^\circ = 12 \times 30^\circ$)? Why did early astronomers consider that there were 12 "zodiacal" signs though actually the Sun intersects 13 constellations? And, further, why did the Babylonian number system have the number of 60 as its base?

Analyzing these questions we find that the four numbers consistently arise: 12, 30, 60, and 360=12x30. In the most ancient calendars originated in the eastern and southeastern Asia a big attention was given to the motions of the Sun, the Moon, and the two largest planets of the Solar system, Jupiter and Saturn. Note that Jupiter make its full revolution around the Sun approximately in 12 years (11.862 years) and Saturn does approximately in 30 years (29.458 years). Based on these numbers, the Ancient Chinese introduced the 60-year cycle of the Solar system during which Saturn makes the two full revolutions around the Sun and Jupiter does the five revolutions.

Ancient scientists were surprised to discover mathematical connection between the main cycles of the Solar system and one of the "Platonic Solids", the dodecahedron. The Egyptians ranked the dodecahedron as the "main Universe figure," symbolizing the "Harmony of the Universe." They thus based their systems of measurement (calendar, time and angle measurement) on the numerical parameters of the dodecahedron (12 faces, 30 edges, 60 plane angles on the dodecahedron surface). These systems were coordinated very well with their "Theory of Harmony" based on the "Golden Proportion" underlying the dodecahedron. According to this hypothesis, the mankind lives for thousands years according to the "Golden Proportion"!

2.5. Striking by Pythagoras. Pythagoras is possibly the most celebrated person in history of science. This name is known to each person studied geometry. "The famous philosopher and scientist, religious and ethical reformer, influential politician, "demigod" in eyes of his followers and "charlatan" under recalls of some of his contemporaries" – such characteristics are given to Pythagoras in the antique literature. The coins with his image exhausted in 430-420 BC testify to exclusive popularity of Pythagoras already at his life. For the 5-th centuries BC it is unprecedented case! Pythagoras was the first among the Greek philosophers who was granted with a special book dedicated to him.



Pythagoras (about 580 B.C. – about 500 B.C.)

His scientific school is internationally known. He organized it in Croton, the Greek colony in the north of Italy. The Pythagorean School or the "Pythagorean Union" was simultaneously both philosophical school, and political party, and religious brotherhood. The status of the “Pythagorean Union" was very severe. Everyone joined the Union should refuse from a personal property in favor of the Union, undertook not to spill a blood, not to eat meat nutrition, to protect secret of their teacher doctrine. It was prohibited to the members of the Union to train other people for reward.

The Pythagorean doctrine touched upon harmony, geometry, number theory, astronomy etc. But most of all the Pythagorean appreciated the results obtained in the theory of harmony because they confirmed their idea: "the numbers determine everything ". Some ancient scientists assure that a concept of the golden section was borrowed by Pythagoras from the Babylonians.

Many great mathematical discoveries were attributed to Pythagoras undeservedly. For example, the famous geometric "theorem of squares" ("Pythagorean Theorem") was known for Egyptians, Babylonians and Chinese’s long before Pythagoras.

In what is a cause of so large Pythagoras popularity already at his life? The answer this question can be given by some interesting facts from his biography placed in the “Biographic dictionary of the persons in the field of mathematics" (1979) [4]. In the article dedicated to Pythagoras it is noticed that *"according to the legend Pythagoras went away to Egypt to be acquainted himself with the wisdom of the eastern scientists and lived there for 22 years. By taking possession of all the Egyptian sciences, including mathematics, he moved to Babylon, where he lived for 12 years and was acquainted himself with the scientific knowledge of the Babylonian priests. The legend ascribes to Pythagoras the visit of India. It is very probably as then Ionia and India had business relations. On returning home (about 530 BC) Pythagoras attempted to organize his philosophical school. However on unknown causes he abandons soon Samos and settles in Croton (the Greek colony in the north of Italy). Here Pythagoras organized the school, which acted almost thirty years".*

Thus, the outstanding role of Pythagoras in development of the Greek culture consists of fulfillment of historical mission of knowledge transmission from the Egyptian and Babylonian priests to the culture of Ancient Greece. Just thanks to Pythagoras who was, without any doubt, one of the most learned thinkers of his time the Greek science received a huge volume of knowledge in the field of philosophy, mathematics and natural sciences, which by getting to favorable medium of the Ancient Greek culture promoted to its rapid development and augmentation.

Developing the idea about Pythagoras' historical role in development of the Greek science, Igor Shmelev in the brochure [2] wrote the following:

"His World renowned name of Pythagoras the Croton teacher obtained after the rite of "consecration". This name is composed from two halves and means the "Elucidated Harmony" because "Pythians" were pagan priests predicted a future in Ancient Greece and Gor personified harmony in Ancient Egypt. So on the decline of their civilization Egyptian priests by transmitting their secret knowledge to the representative of gained forces civilization cemented symbolically in one person the Union of man's and woman's origins, the bastion of Harmony ".

2.6. Golden Section in the Greek Art. The idea of harmony based on the golden section became one of the fruitful ideas of the Greek art. A nature taken in a broad sense included also of the person creative patterns, art, music, where the same laws of rhythm and harmony act. Let us give a word to Aristotle:

"The Nature aims to contrasts and from them, instead of from similar things, forms consonance ... It combined a male with a female and thus the first public connection is formed through the connection of contrasts, instead of by means of similar. As well the art, apparently, acts in the same way by imitating to the nature. Namely the painting makes the pictures conforming the originals by mixing white, black, yellow and red paints. Music creates unified harmony by mixing different voices, high and low, lingering and short, in congregational singing. The grammar creats the whole art from the mixture of vowels and consonants".

To take a material and to eliminate all superfluous is the aphoristically embodied artist schedule. And this is the main idea of the Greek art, for which the "golden section" became some aesthetic canon.

Theory of proportions is the basis of art. And, certainly, the problem of proportionality could not pass past Pythagoras. Among the Greek philosophers Pythagoras was the first one who attempted mathematically to understand an essence of musical harmonic proportions. Pythagoras knew that the intervals of the octave can be expressed by numbers, which correspond to the certaing oscillations of the cord, and these numerical relations were put by Pythagoras in the basis of his musical harmony. Knowledge of arithmetical, geometrical and harmonic proportions, and also the law of the "golden section" are attributed to Pythagoras. Pythagoras paid a special, outstanding attention to the "golden section" by making the pentagon or pentagram as distinctive symbol of the "Pythagorean Union".

Plato used the five regular polyhedrons ("Platonic Solids") and emphasized their "ideal" beauty by borrowing the Pythagorean doctrine about harmony. Importance of proportions is emphasized by Plato in the following words:

„Two parts or values cannot be satisfactorily connected among themselves without third part; the most beautiful link is that, which gives the perfect unit together with two initial values. It is reached in the best way by proportion (analogy), in which among three numbers, planes or bodies, the mean one so concerns to the second one, as the first one to the mean one, and also the second one to the mean one as the mean one to the first one. This implies, that the mean one can exchange the first one and the second one, the first one and the second one can exchange the mean one and all things together thus makes a indissoluble unit ".

As the main requirements of beauty Aristotle puts forward an order, proportionality and limitation in the sizes. The order arises then, when between parts of the whole there are definite ratios and proportions. In music Aristotle recognizes the octave as the most beautiful consonance taking into consideration that a number of oscillations between the basic ton and the octave is expressed by the first numbers of a natural series: 1:2. In poetry, in his opinion, the rhythmic relations of a verse are based on small numerical ratio; thanks to this it is reached a beautiful impression. Except simplicity based on commensurability of separate parts and the whole, Aristotle as well as Plato recognizes the highest beauty of the regular figures and proportions based on the "golden section".

The three adjacent numbers from the initial fragment of Fibonacci series: 5, 8, 13 are values of differences between radiuses of circumferences lying in the basis of the schedule of construction of the majority of Greek's theatres. The Fibonacci series served as the scale, in which each number corresponds to integer units of Greek's foot, but at the same time these values are connected among themselves by unified mathematical regularity.

At construction of temples a man is considered as a "measure of all things": in the temple he should enter with a "proud raised head ". His growth was divided into 6 units (Greek foots), which were marked on the ruler, and they were connected closely to the sequence of the first six Fibonacci numbers: 1, 2, 3, 5, 8, 13 (their sum is equal to $32=2^5$). By adding or subtracting of these standard line segments necessary proportions of building are reached. A six-fold increase of all sizes, laying aside of the ruler, saved a harmonic proportion. Pursuant to this scale also temples, theatres or stadiums are built.

The magnificent Parthenon

The ancient Greeks kept to us magnificent architectural monuments delivering modern people the same aesthetic enjoying as well as their far ancestors. And among them the first place rightfully belongs to Parthenon. A construction of Parthenon is connected to dramatic period of the Ancient Greece history. In 480 BC the Persian army intruded in Greece. Hordes of the barbarians moved from the North and stayed near to the Fermopil gorge. The 300 Spartanian soldiers covering a withdrawal of the main troops blocked their path. As a result of betraying all of them were killed together with their leader, the king Leonid. The Persian army trapped and routed Athens. But Greeks with honor maintained high-gravity trial by routing the Persian fleet and army. The victory of the Greeks over Persians meant celebration of principles of democracy and freedom; it resulted in a new fruitful impulse in Greek's art, to the epoch of high classics. In art's works of this period feelings of majesty and pleasure dominate. The forms of art's works distinguished by a high harmony, plastics, humanism.

The Athens temple of Parthenon, the magnificent construction of the Athenian Acropolis is implementation of these qualities. During 15 years of Pericle's government the temples, altars, sculptures unusual on their beauty were constructed in Athens. The outstanding Greek's sculptor Fidij was the chief of all art's works. All the second half of the 5^{th} century BC at the Acropolis there was built the temples, altar and statue of Athens-Warrior. In 447 BC the building of the Athens temple of Parthenon began and it continued until 434 BC. For creation of harmonic composition on the "sacred hill" its builders increased the hill by building the powerful embankment. The modern researchers established that the length of the hill before Parthenon, the lengths of the Athens temple and of the Acropolis segment behind of Parthenon are in proportion of the golden section! Thus, the golden proportion was used already at the creation of composition of the temples on the "sacred hill".

As outcome of joined efforts of architects, sculptures and all people of Ancient Greece was the creation of the goddess Athens temple, the "magnificent Parthenon" (Fig.13), which rightfully is considered as the greatest monument of the Ancient Greek architecture.

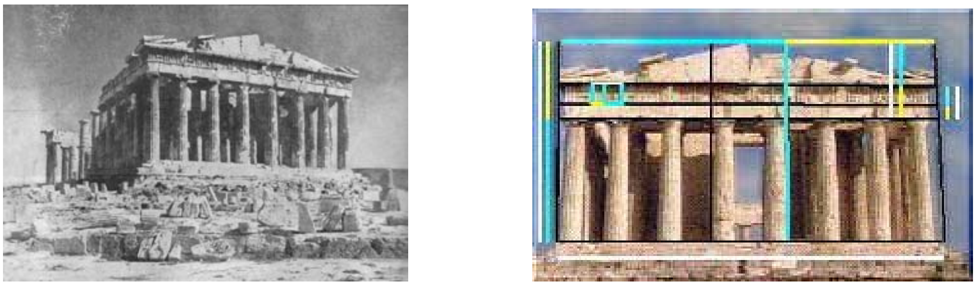


Figure 13. Parthenon and its harmonic analysis

The harmonic analysis of Parthenon was carried out by many researchers. And though these researches differ a little by approaches, but all they agreed in the main: Parthenon distinguishes itself by surprising grandeur and steep humanity of architectural and sculptural images and the main cause of Parthenon's beauty is the exclusive harmony of its parts based on the golden proportion.

2.7. Icosahedral-Dodecahedral structure of the Universe. The great Greek's philosopher Plato (427-347 BC) was the second (after Pythagoras) scientific figure contributed into development of the idea of the Universe harmony. According to the remark of the commentator of the last issuing of Plato's works for him "*all cosmic proportionality is based on the principle of the "golden" or harmonic proportion"*.

Plato's cosmology is based on the regular polyhedrons called "Platonic Solids". Each Plato's Solid symbolized some of the five "beginnings" or "elements": *tetrahedron* – a body of fire, *octahedron* – a body of air, *hexahedron* (cube) – a body of the Earth, *icosahedron* – a body of water, dodecahedron – a body of the Universe. A representation about the "throughout" harmony of the Universe was associated invariably with its embodiment in these regular polyhedrons. And the fact that the dodecahedron, the main "cosmic" figure, was based on the golden proportion gave to the latter significance of the main proportion of the Universe.

"Euclid did not intend to write the systematic tutorial on geometry. He set as a goal to write the text-book about the regular polyhedrons intended for the beginners by virtue of what he set forth all necessary information" – the joke of the known English geometer d'Arci Thompson, as well as any good sharpness, contains in itself a grain of true. According to Proclus, the commentator of the "Euclidean Elements", Euclid considered the methods of construction of the Platonic Solids as the "crown" of all the thirteen books of the "Elements". And he placed just this major mathematical information in the concluding, thirteenth book.

Plato's cosmology became the basis of the so-called icosahedral-dodecahedral doctrine, which by red thread passes through all human science. The essence of this doctrine consists of the fact that the dodecahedron and the regular icosahedron are typical forms of Nature in all its manifestations starting since cosmos and ending by micro-cosmos.

The problem about the form of the Earth permanently took minds of scientists since antique times. And when the hypothesis about the spherical form of the Earth got scientific grounds there arose an idea the Earth represents by itself the dodecahedron by its form. So, already Plato wrote: *"The Earth, if to look at it from above, is similar to the ball consisting of 12 skin's pieces"*. This Plato's hypothesis found further scientific development in the works of physicists, mathematicians and geologists. So, the French geologist de Bimon and the well-known mathematician Poincare considered the form of the Earth represents by itself the deformed dodecahedron.

The Russian geologist Kislitsin also used in his researches the idea about the dodecahedral form of the Earth. He put forward the hypothesis that 400-500 millions years ago the geo-sphere of the dodecahedral form began to turn into the geo-icosahedron. However, such transformation appeared not full and uncompleted. As the result the geo-dodecahedron appeared to be inscribed into the frame of the geo-icosahedron.

Last years the hypothesis about the icosahedral-dodecahedral form of the Earth was subjected to verification. For this purpose scientists combined the axis of the dodecahedron with the axis of the terrestrial globe and, gyrating around of it this polyhedron, paid attention that its edges coincide with gigantic disturbances of Earth's crust. By taking then regular icosahedron they established that its edges coincide with more small-sized partitionings of Earth's crust (mountain ranges, breaks etc.). These observations confirm the hypothesis about proximity of the tectonic framework of Earth's crust with the forms of the dodecahedron and icosahedron. All these examples confirm surprising insight of Plato's intuition.

A long time it was considered, that in the inorganic nature there were not used almost dodecahedron and regular icosahedron having so-called "pentagonal" symmetry axis, but the "pentagonal" symmetry axis is a constant "satellite" of the living nature. The regular icosahedron is geometric object, which form is used by viruses, that is, the icosahedral form and pentagonal symmetry are fundamental in organization of living material.

The discovery of the quasi-crystals (based on the regular icosahedron) made in 1984 by the Israel physicist Dan Shechtman (in more details about this discovery we will tell later) became the outstanding event in modern physics, as this discovery showed that the "pentagonal" symmetry and the icosahedral form play also a fundamental role in crystallography.

2.8. Leonardo Pisano Fibonacci. The "Middle Ages" in our consciousness associate with inquisition orgy, campfires, on which witches and heretics are incinerated, and also with crusades for "the body of God". Science in those times obviously was not "in centre of social attention". At these conditions the appearance of the mathematical book "Liber abaci" ("the book about an abacus") written in 1202 by the Italian mathematician Leonardo Pisano (by the nickname of Fibonacci) was the relevant event in the "scientific life of society".

Who was Fibonacci? And why his mathematical works are so important for the West-European mathematics? To answer these questions it is necessary to reproduce the historical epoch, in which Fibonacci lived and worked.

It is necessary to note that the period since the 11th until the 12th centuries was the epoch of brilliant flowering of the Arabian culture but at the same time the beginning of its downfall. In the end of the 11th century, that is, in the beginning of the Crusades the Arabs were, doubtlessly, the most educated people in the Glob surpassing in this respect of their Christian enemies. Jet before the Crusades the Arabian influence began to penetrate to the West. However the greatest infiltration of the Arabian culture to the West began after the Crusades weakened the Arabian world, but on the other hand, boosted the Arabian influence on the Christian West. Not only the Palestinian cotton and sugar, pepper and black wood of Egypt, self-color rocks and specifies of India are searched and appreciated by the Christian West in the Arabian world. He starts to assess properly the cultural heritage "of the great antique East". The world discovered by the West researchers blinded them by the brilliant Arabian art works and scientific achievements and therefore there arose in the West world demands on the Arabian geographical maps, tutorials on algebra and astronomy, and the Arabian architecture.

The emperor Fridrich Gogenstaufen, the apprentice of the Sicilian Arabs and the admirer of the Arabian culture, was one of the most interesting persons of the Crusades epoch, the harbinger of the Renaissance epoch. The greatest European mathematician of the Middle Ages Leonardo Pisano (by the nickname of Fibonacci that means the son of Bonacci) lived and worked at his palace in Pisa.

Leonardo Pisano Fibonacci (about 1170 – after 1228)

About Fibonacci life it is known a little. Even the exact date of his birth is obscure. It is supposed that Fibonacci was born in the eighth decade of the 12th century (presumptively in 1170). His father was a merchant and a government official, the representative of the new class of the businessmen generated by the "Commercial Revolution". In that time the city of Pisa was one of the largest commercial Italian centers actively cooperating with the Islam East, and Fibonacci's father traded in one of the trading posts founded by Italians on the northern coast of Africa. Thanks to this circumstance he could give his son, the future mathematician Fibonacci, good mathematical education in one of the Arabian educational institutions.

One of the known historians of mathematics Moris Cantor called Fibonacci *"as the brilliant meteor flashed past on the dark background of the West-European Middle Ages"*. He supposed that, probably, Fibonacci perished during one of the Crusades (presumptively in 1228) accompanying the emperor Fridrich Gogenstaufen.

Fibonacci wrote several mathematical works: "Liber abaci", "Liber quadratorium", "Practica geometriae". The book "Liber abaci" is the most known of them. The problem of "rabbits multiplying" is the most known among the different mathematical problems formulated by Fibonacci. This one resulted in discovery of the numerical sequence 1, 1, 2, 3, 5, 8, 13... called *Fibonacci numbers.*

Though Fibonacci was one of the brightest mathematical minds in the history of the West-European mathematics, however his contribution to mathematics is belittled undeservedly. A significance of Fibonacci's mathematical creativity for mathematics is assessed properly by the Russian mathematician Prof. Vasiljev in his book "Integer Number" (1919):

"The works of the learned Pisa's merchant were so above the level of mathematical knowledge even of the scientists of that time, that their influence on the mathematical literature became noticeable only in two centuries after his death namely in the end of the 15th century, when many of his theorems and problems were included by Leonardo da Vinci's friend, professor of many Italian universities Luca Pacioli in his works and in the beginning of the 16th century,

when the group of the talented Italian mathematicians: Ferro, Cardano, Tartalia, Ferrari gave the beginning of higher algebra by the solution of the cubical and biquadrate equations".

It follows from this statement that Fibonacci almost for two centuries anticipated the West-European mathematicians of his time. Like to Pythagoras who obtained his "scientific education" in the Egyptian and Babylonian science and then promoted transferring the obtained knowledge to the Greek science, Fibonacci obtained his mathematical education in the Arabian educational institutions and many from the obtained there knowledge, in particular, the Arabian-Hindu decimal notation, he attempted to introduce to the West-European mathematics. And like to Pythagoras a historical role of Fibonacci for the West science consists of the fact that he by his mathematical books promoted transfering the Arabian mathematical knowledge to the West-European science and by that he created fundamentals for further development of the West-European mathematics.

2.9. Fibonacci numbers. The golden section relates closely to so-called Fibonacci numbers [5, 6] discovered in the 13th century by the famous Italian mathematician Leonardo Pisano Fibonacci while solving the problem of "rabbits multiplying".

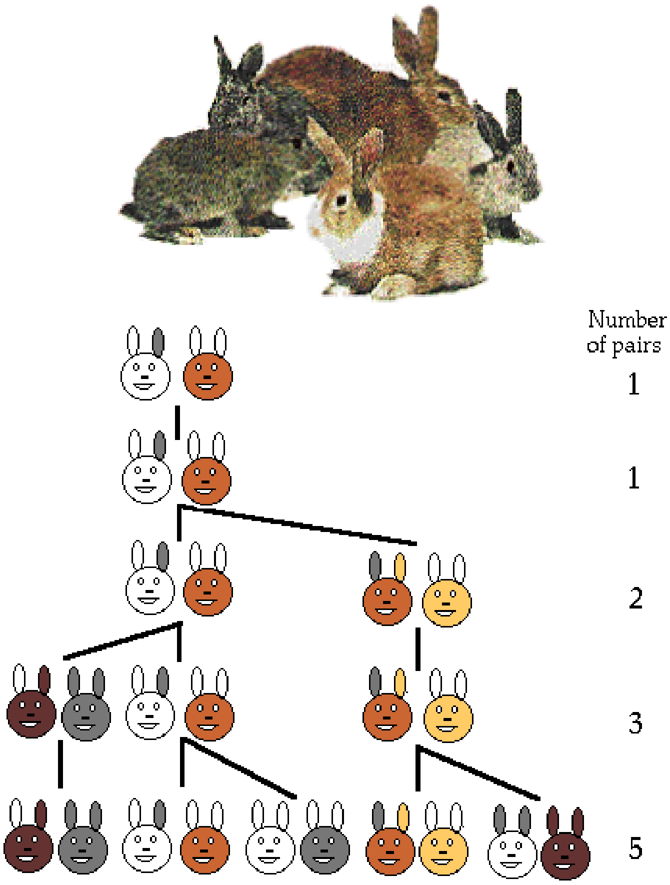


Figure 14. Fibonacci's rabbits

Let us assume that *A* and *B* are the pairs of "mature" and "baby" rabbits respectively. A rule of "rabbit multiplying" consists of realization of the following month's passage: ***B→A*** (maturing the "baby" pair); ***A→AB*** (birthing the "baby" pair).

A statement and solution of the problem is considered to be Fibonacci's great contribution to combinatorial analysis. While solving the problem Fibonacci discovered the first recursive formula in mathematics history

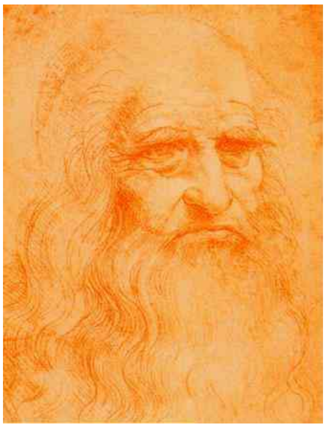
$$F_n = F_{n-1} + F_{n-2} \quad (7)$$

generating Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, … at the initial conditions

$$F_1 = F_2 = 1. \quad (8)$$

Note that by this discovery the Italian mathematician Leonardo Pisano Fibonacci anticipated the method of recurrence relations, which was regarded as the most appropriate for solving combinatorial problems.

2.10. The idea of harmony in the Renaissance epoch. The Renaissance epoch in history of culture of the Western and Central Europe countries is the transient epoch from the medieval culture to the culture of the New Time. The most typical feature of this epoch is humanistic world outlook and reversal to the antique cultural heritage, as though the "Renaissance" of the ancient culture. The Renaissance epoch is distinguished by large scientific shifts to the field of natural sciences. A close connection to art was a specific feature of the Renaissance epoch science and this feature is expressed sometimes in creativity of some learned persons of the Renaissance epoch. Leonardo da Vinci, the greatest artist, scientist, inventor, and engineer, was the brightest example of such “multilateral” person.



Leonardo da Vinci (1452 - 1519) and his famous picture “Joconda”

Together with other achievements of the ancient art the scientists and artists of the Renaissance epoch perceived with great enthusiasm the Pythagorean idea of the Universe harmony and the golden section. And it is not incidentally that just Leonardo da Vinci introduced in wide use the name of the "golden section", which at once became the aesthetic canon of the Renaissance epoch.

The idea of harmony appeared among those antique conceptual ideas, to which the church treated with a great interest. According to the Christian doctrine, the Universe is God’s creation and therefore is subjected implicitly to his will. And at the creation of the Universe the Christian God was guided by the mathematical principles. In science and art of the Renaissance epoch this catholic doctrine gained the form of searching the mathematical schedule used by the God at the creation of the Universe.

In opinion of the modern American historian of mathematics Moris Klain, just a close union of the religious doctrine about the God as the creator of the Universe and the antique idea about numerical harmony of the Universe became one of the major causes of huge cultural splash of the Renaissance epoch. Brightly the main purpose of the Renaissance epoch science is expressed in the following Kepler's words:

"The main purpose of all researches of the external world should be discovering the rational order and harmony sent by the God to the World and expressed by him by using mathematics language".

The same idea, the idea of the Universe harmony, the expression of its ordering and perfection is transformed into the main idea of the Renaissance art. In works of Bramante, Leonardo da Vinci, Rafael, Jordano, Tizian, Alberti, Donatello, Michelangelo it was shown the strong ordering and harmony subordinated to the golden proportion. The law of harmony, law of a number is uncovered in the art works and scientific-methodical researches by Leonardo da Vinci, Durer, Alberti.

In the famous portrait of Mona Lisa ("Jokonda"), which was completed by Leonardo da Vinci in 1503, the image of rich townswoman is presented as implementation of a raised feminist ideal, not losing thus of intimate-human charm (the famous "Jokonda's smile"); a relevant composition element becomes the cosmically vast landscape running in the cold mist. The picture of the ingenious artist attracted attention of researchers and they found out the composition construction of the picture is based on the two "golden" triangles, which are the parts of the "pentagram".

In the period of the Italian Renaissance researches in the field of the proportionality theory of sculpture and architecture works are continued. In this period in Italy the works of the famous Roman architect Vitruvij were reissued and they rendered a certain influence on the works of the Italian art theorists (Alberti). By arising in Florence, the classic style of the High Renaissance created the sculpture and architecture monuments in Rome, Venice and other cultural centers of Italy.

Apart from the artists, architects and sculptures of this epoch all musical culture develops under strong influence of the antique harmony ideas. In this period the 12-sound musical system was introduced into music by the well-known philosopher, physicist and mathematician Mersenn. In some of his works ("Tractate about the Overall Harmony", "The Overall Harmony") Mersenn considers music as the integral part of mathematics and sees in it and in its consonance sounding one of the main ways of global harmony and beauty development.

Just in this period the first book dedicated to the "golden section" appears. The famous Italian mathematician and learned monk Luca Pacioli is the writer of the book.

2.11. Luca Pacioli. The culture of Ancient Greece and the culture of Rome and Byzantium are the two powerful flows of spiritual values, which gave sprouts of the new Renaissance culture and became a cause of the origin of the Renaissance epoch "Titans". The "Titan" is the most precise word with respect to such persons, as Leonardo da Vinci, Michelangelo, Nicola Copernicus, Albert Durer and others. And a honor place among them belongs to Luca Pacioli, the outstanding Italian mathematician of the Renaissance epoch [7].

He was born in 1445 in provincial town Borgo San-Sepolcoro that in translation from Italian sounds is not too joyous: "City of the Sacred Coffin".

We do not know how much years was to the future mathematician when he began to study at the workshop of the artist Pierro della Franchesko. Franchesko's glory "rattled" in all Italy. This was the first meeting of juvenile talent with a great person. Pierro della Franchesko was the artist and mathematician, but only the second vocation of his teacher found out a respond in heart of his schoolboy. Juvenile Luca, the mathematician from the God, was loved in number world; the numbers were perceived by him as some universal key simultaneously opening an access to the true and to the beauty.

Leone Battista Alberti, the famous Italian architect, scientist, writer, and musician, was the second great person met on the life path of Luca Pacioli. The following Alberti's words entered deeply to Luca's consciousness:

"A beauty is a certain consent and consonance of the parts in the whole, which parts they are by, answering to stringent number, limitation and arrangement, which are demanded on harmony, that is, on the absolute and primary beginning of the nature "

Being loved in the number world and following to Pythagoras Luca Pacioli will repeat that numbers underlay the Universe.

In 1472 Luca Pacioli becomes by the monk of the Franciscan order that gives him a possibility to be engaged in science. The events showed that he made a right decision. In 1477 he gets a Professor chair at Perugi University.

Diuina proportione

Opera a tutti glingegni perspicaci e curiosi necessaria Oue ciascun studioso di Philosophia: Prospectiua Pictura Sculptura: Architectura: Musica: e altre Mathematice: suauissima: sottile: e admirabile doctrina consequira: e delectarassi: cõ varie questione de secretissima scientia.

M. Antonio Capella eruditiss. recensente: A. Paganius Paganinus Characteribus elegantissimis accuratissime imprimebat.

Luca Pacioli and his famous book "Divina Proportione"

The following portrait's description of Luca Pacioli gives a presentation about him:

"The beautiful, vigorous young man: the raised and rather broad shoulders expose inherent physical force, a potent neck and developed jaw, expressive face and eye beaming nobleness and intellect, underline force of nature. Such professor could urge to listen himself and to respect his subject".

Pacioli combines successfully his pedagogical work with scientific activity: he starts to write the encyclopedic mathematics book. In 1494 his book was issued under the title *"The Sum of arithmetic's, geometry, doctrine about proportions and relations"*. All material of the book is divided into two parts; the former part is dedicated to arithmetic and algebra, the latter one to geometry. One of the book sections is dedicated to problems of mathematics application to commercial business and in this part his book is a development of the famous Fibonacci's book "Liber abaci" (1202). In essence, this mathematical work by Luca Pacioli was a total of the mathematical knowledge of the Renaissance epoch.

Fundamental Luca Pacioli's book promoted to his glory. When in 1496 in Milan, the largest city and state of Italy, the mathematics chair was opened Luca Pachioli was invited to take this chair.

In that period Milan was the center of science and art and many outstanding scientists and artists lived and worked there. And one of them was Leonardo da Vinci who became the third great person met on the life path of Luca Pacioli. Under direct Leonardo da Vinci's influence he starts to write his second great book "De Divine Proportione"

This book published by Pacioli in 1509 rendered noticeable influence on his contemporaries. Pacioli's folio was one of the first brilliant examples of the Italian book-printing art. A historical significance of the book consists of the fact that it was the first mathematical book dedicated to the "golden section". The book is illustrated by 60 (!) magnificent figures executed by Leonardo da Vinci. The book consists of the three parts: in the first part the properties of the "golden section" are given; the second part is dedicated to regular polyhedrons; the third one is dedicated to applications of the golden section in architecture.

Appealing to the "State", "Laws", and "Timey" by Plato Luca Pacioli sequentially deduces the 12 (!) different properties of the golden section. Characterizing these properties, Pacioli uses a rather strong epithets: "exclusive", "remarkable", "almost supernatural" etc. Uncovering the given proportion as the universal relation expressing a perfection of beauty in Nature and Art he names it the "Divine Proportion" and considers it as the "instrument of thinking", the "aesthetic canon", and the "Main Principle of the Universe".

This book is one of the first mathematical works, in which the Christian doctrine about the God as the creator of the Universe gets a scientific substantiation. Pacioli names the golden section as the "Divine Proportion" and selects a number of the golden proportion properties, which, in his opinion, are generic to the God himself. Following Plato's "Timey" he selects the dodecahedron as the main figure of the Universe.

In 1510 Luca Pacioli was 65 years old. He was tired and grown old. In the library of Bolon University the manuscript of the unpublished Pacioli book "About forces and quantities" is stored. In the foreword we found the sad phrase: *"The last days of my life approach"*. He died in 1515 and was buried at the cemetery of his native city Borgo San-Sepolcoro.

After his death the works of the great mathematician were forgotten almost for four centuries. And when at the end of the 19th century his works became internationally known, the grateful offsets after 370-year's oblivion put the monument on his grave with the following inscription:

"To Luca Pacioli who was the friend and adviser of Leonardo da Vince and Leone Battista Alberti, who was the first scientist given to algebra a language and a frame of science, who applied his great discovery to geometry, who invented the double accounting and who gave in his mathematical works fundamentals and invariable standards for succeeding generations".

2.12. Johannes Kepler: from "Mysterium" to "Harmony". Johannes Kepler is known for all educated mankind as the author of the three famous astronomical laws inverted the astronomical ideas existed from the antique times. But it is less known that these laws were obtained by Kepler as the partial outcomes of his grandiose program on research of the Universe harmony put forward by him in young age.

Johannes Kepler (1571 - 1630)

Johannes Kepler was born in 1571 in poor protestant family. In 1591 he enrolled in the Tubingen Academy where he got a quite good mathematical education. Just there the future great astronomer was acquainted with the heliocentric system by Nicola Copernicus. After graduation of the Academy Kepler obtained a Master degree and then was appointed as mathematics teacher in the Graz High School. The small book with the intriguing title "Misterium Cosmographium" published by Kepler in 1596 in the age of 25 years was his first astronomical work.

Reading this first Kepler's book it is impossible to be not surprised by his imaginations. A deep belief in existence of the Universe harmony imposed an impress on all Kepler's thinking.

The purpose of his researches was formulated by Kepler as the following:

"The kind reader! In this book I intended to demonstrate, that the all-good and almighty God at the creation of our moving world and at the arrangement of the celestial orbits used the five regular polyhedrons, which from Pythagoras and Plato's times and up to now got so loud glory, and selected a number and proportions of celestial orbits, and also the relations between the planet motions pursuant to the nature of the regular polyhedrons.

Especially I interested by the nature of the three things: why the planets are arranged so but no otherwise, namely, a number, sizes and motions of celestial orbits ".

So, already in the foreword to his first book the 25-year's Kepler put forward the problem being the main problem of the new time physics, the problem of natural phenomena causes. So natural today, this problem during Kepler's times sounded unusually. In Ptolomey's and even in Copernicus astronomy this problem did not be formulated. Following to this old tradition, the astronomers considered a problem of their science only in possible precise description of planet's motions and prediction of celestial phenomena.

How Kepler answered the surprising problems? After check of numerous hypotheses connected with arrangement of planets Kepler came to the following geometrical model of the Solar system based on the "Platonic Solids":

"The Earth orbit is the measure of all orbits. Around of it we circumscribe the dodecahedron. The orbit circumscribed around of the dodecahedron is the Mars orbit. Around of the Mars orbit we circumscribe the tetrahedron. The orbit circumscribed around of the tetrahedron is the Jupiter orbit. Around of the Jupiter orbit we circumscribe the cube. The sphere circumscribed around of the cube is the Saturn orbit. In the Earth orbit the regular icosahedron is inserted. The orbit entered in it is the Venus orbit. In the Venus orbit the octahedron is inserted. The orbit entered in it is the Mercury orbit".

Kepler figured his model geometrically as the following (Fig. 16).

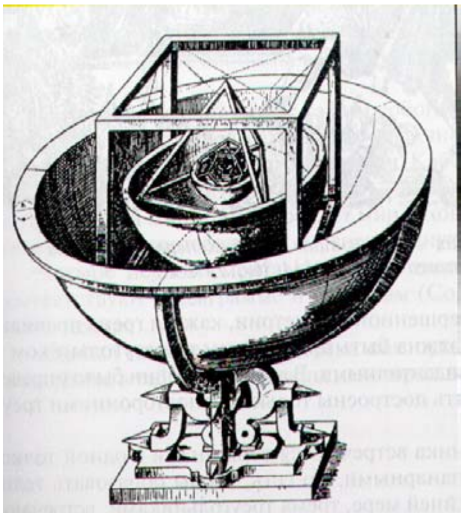


Figure 15. Kepler's "Cosmic Cup" as a model of the Solar system

In Kepler's opinion, a secret of the Universe consists of the following: the Universe is arranged on the basis of the unified geometrical principle! But Kepler's joy was prematurely. In spite of his exaltation character Kepler had all capacities of serious scientist. Kepler perceived that the theory should be coordinated with observations data. By constraining his delight Kepler undertakes to check his model.

The unified geometrical principle allowed to Kepler to give the answer on the two of the three problems put forward by him: (1) to explain a number of the known then planets (with the help of the five "Platonic Solids" it is possible to construct 6 orbits; it follows from here a conclusion about existence of the six planets known in that period); (2) to give the answer to the problem about distances between planets.

The answer to the third problem (about motion of planets) appeared by most difficult and its solution was obtained by Kepler many years after.

Kepler's model was based on the supposition about spherical nature of planet's motion. By obtaining to his disposal the data of perennial observations of the famous astronomer Brage and by making his own observations, Kepler made convinced of necessity to reject astronomical constructions as his forerunners, Ptolomey and Copernicus, and his own one's. Carefully observing the planet orbits he came to the following conclusion:

"The fact that the planet motions are circular is testified by their incessant repeatability. The mind extracting this true from experience at once concludes from here, that the planets are rotated on the "ideal" circles, wherefore among the plane figures the circle and among the spatial figures the celestial orbit are considered as perfect. However at the more close examination it appears that the experience learns a little bit other, namely: planet's orbits differ from simple circles ".

The tireless search of the laws corresponding to the observational data ended by the discovery of the three famous laws of planet's motion. The former two laws were stated by Kepler in the book "New Astronomy" published in 1609.

The first Kepler's law introduced the *ellipse* as geometric model of planet's motion in contradictory to the astronomical tradition. The first Kepler's Law asserts that *planets move on ellipsoidal orbits*. But it keeps silent about a question how fast the planets are moved on their orbits. The answer this question is given by the second Kepler's Law, which asserts: *"The areas covered for equal time by the segment, draw from the Sun to the planet are equal".*

But one relevant problem remained unsolved. Under what law do the distances from the Sun to planets change? The matter was complicated by the fact that the distances from the planets to the Sun are non-constant and Kepler attempts to grope for a new principle for the solution of this complicated problem. Here again the philosophical Kepler's beliefs ascending to Pythagoras and Plato came on the aid. According to Kepler's deep belief the God created the nature on the basis not only mathematical but also harmonic principles. He believed in "music of spheres", which charming melodies are embodied not in sounds but in planet's motions capable to bear harmonic consonances. Following to this idea, Kepler by the way of surprising combinations of the mathematical and musical nature arguments Kepler came to the third law of planet's motion, which asserts: *"If T is a cycle time of the planet circulation around of the Sun, and D is its middle distance from the Sun, then we have:*

$$T^2 = kD^3,$$

where k is a constant equal for all planets".

But there came old age. The death (in 1630) interrupted Kepler's work under the last book "Somnium" ("Dreaming"), the first scientific-fantastic novel about flight on the Moon. But about harmony was not written any more word. There were no captious checks; there were no new hypotheses. Kepler got tired: *"My brain gets tired, when I attempt to understand, that I have written, and it is difficult to me already to reestablish connection between figures and text, which was found by me at one time ...".*

With Kepler's death his discoveries were forgotten. Even wise Descartes does not know about Kepler's works. Galilei had not found necessary to read his books. Only for Newton Kepler's Laws found new life. But the harmony was not be interested for Newton. He had the Equations. There came the New Times.

Kepler terminated the epoch of "scientific romanticism", the epoch of harmony and golden section that was inherent in the Renaissance epoch. But on the other hand, his scientific works became the beginning of a new science, which began to develop since the works of Descartes, Galilei and Newton.

But with Kepler's death the golden section considered by him as one of the "geometry treasures" was forgotten. And this strange oblivion was continued almost for two centuries. The interest in the golden section is revived again only in the 19^{th} century.

2.13. Zeising's Law of Proportionality. In the 19th century a large contribution to development of the theory of proportionality was made by the German scientist Zeising issued the book "Neue Lehre von den Proportionen des menschlichen Korpers" (1854). This one is until now by widely quoted book among the works dedicated to proportionality problem.

Outgoing from the fact that proportion is the ratio of the two unequal parts between themselves and to the whole in their perfect combination Zeising formulated the "Law of Proportionality" as the following:

"*The division of the whole on the unequal parts looked proportional when the ratio of parts of the whole between themselves is the same that the ratio of them to the whole, that is, the ratio given by the golden section* ".

Attempting to prove that the Universe is subjected to this law Zeising tries to find it both in the organic and in the inorganic world.

To confirm this he gives diverse data about ratios of mutual distances between themselves of celestial heavenly bodies (corresponding to the golden section), also he finds the same ratio in the constitution of human body, in the configuration of minerals, in plants, in the sound chords of music, and in the architectural monuments. By considering of Apollo and Venus statues Zeising finds that at division of the human altitude by the given ratio the line of division passes through natural partitioning of the human body. The first division passes through the navel, the second one through the middle of the neck etc., that is, all sizes of the separate body parts are obtained by the division of the whole in the golden section.

Analyzing significance of the golden section law in music Zeising shows that the ancient Greeks attributed an aesthetic impression of chords to proportional division of the octave through the arithmetical and harmonic proportion. Basing on the fact that only those combinations of tones are beautiful when they are in proportional ratio and that the combination of only two tones does not give a full harmony, Zeising shows that the most pleasant consonances have such combination of tones when the ratio of frequencies included in the chord is close to the golden proportion. For example, the combination of small third with the octave of the main tone corresponds to frequency ratio: 3:5; the combination of the large third with the octave of the main tone gives the frequency ratio: 5:8 (note that the numbers 3, 5, 8 are Fibonacci numbers!). Further Zeising makes a conclusion as these two-tone combinations among two-valued combinations are the most pleasant for hearing, it, apparently, explains the fact why only these combinations finish musical periods. By using this fact Zeising explains why impromptu national melody and simple music of the two French or English horns is gone in sixths and their supplements, the thirds.

Zeising pays attention for one curious fact. As is known, the major (man's) and minor (woman's) harmonies are constructed on the basis of the major and minor triad. The major triad constructed on the basis of the large third is a fine consonance since acoustical point of view. This one creates the impression of balance, physical perfection, light, vigor integrated in life with a concept of "majority". The minor triad constructed on the basis of the small third is a consonance, which is incorrect since acoustical point of view. This one creates the impression of the broken sounding and has a nature of gloominess, sadness, weakness integrated in life with a concept of "minority". In this connection Zeising notes that the combination of the octave with the large third of the main tone corresponds to the ratio of the lower and upper parts of man's body, and the combination of the octave with the small third of the main tone corresponds to the ratio of the lower and upper parts of woman's body.

Passing to a significance of the law of proportionality in architecture Zeisung shows that the architecture in the field of arts takes the same place as well as the organic world in the nature inspiring the inert matter on the basis of world's laws. Systematization, symmetry and proportionality thus are its indispensable attributes; it follows from here that the problem of proportionality laws stands considerably more acute in architecture, than in sculpture or in painting.

2.14. Fechner's Experiments. The "psychological experiments by Fechner", one of the authors of the main psychophysical law, got a wide notoriety in that period. "Fechner's experiments" were directed on revelation of feeling of beauty and harmony for adult people. To evaluate aesthetic feelings the 10 white rectangles with the ratio of sides from 1:1 (the square) up to 2:5 were presented to all the participants of "Fechner's experiments" (228 men and 119 women). The "golden" rectangle with the side ratio 21:34 was one of them.

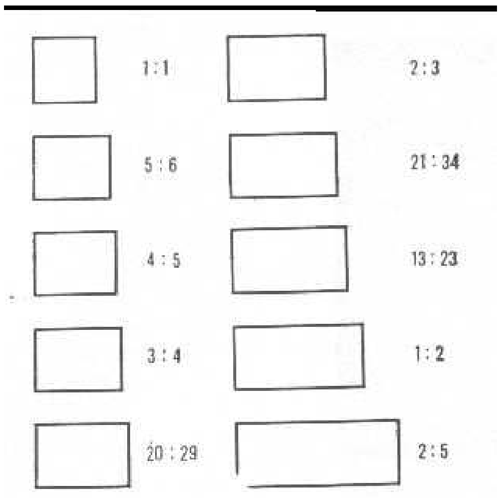


Figure 16. Fechner's rectangles

By means of comparison it was necessary to put in order the compared rectangles by selecting one of the rectangles, which is most preferable since aesthetical point of view. The experiments appeared by highly favorable for the "golden" rectangle 21:34.

In 1958 the English scientists repeated «Fechner's experiments». These experiments again appeared by rather favorable for the "golden" rectangle. The majority of participants (35%) immediately indicated the "golden" rectangle with the side ratio: 21:34. The rectangles with the ratios 2:3 and 13:23 (adjacent to the "golden" rectangle) also were estimated rather highly (20% - fir the former case and 19 % - for the latter one). All remaining rectangles got no more than 10%.

The same experiments made in children's audience gave other results. It was made from here the conclusion that, apparently, the feeling of the beautiful in its most thin and steep parties is generic only for mature persons.

«Fechner's experiments» explain why we prefer often the shape of "golden" rectangle in the rectangular everyday household items (books, matchboxes, etc.). Gary Meisner attracts our attention in his remarkable web site (http://goldennumber.net/stocks.htm) that the USA credit cards are in the shape of the "golden" rectangle.

Figure 17. Credit card

Thus, the science of the 19th century returned again to search of the answer to those "eternal" problems, which were put forward still by the Ancient Greeks. There was ripened the belief that the "universal law" of a number and rhythm expressing its structural and functional parties prevails in the Universe. In this connection the interest in the "golden section" is awaked again in mathematics of the 19th century.

2.15. Lucas numbers. Above we introduced into being *Fibonacci numbers* given by the recursive formula of (7) at the initial conditions of (8). Note the values of integers generated by the recursive formula of (7) depend on the initial conditions. In addition to Fibonacci numbers there exist so called *Lucas numbers* L_n 1, 3, 4, 7, 11, 18, 29, 47, … generated by the same recursive formula

$$L_n = L_{n-1} + L_{n-2} \tag{9}$$

at the other initial conditions:

$$L_1 = 1;\ L_2 = 3 \tag{10}$$

The F_n - and L_n - sequences defined for the discrete values of n in the range of $-\infty$ to $+\infty$ are given in Table 1.

Table 1

n	0	1	2	3	4	5	6	7	8	9	10
F_n	0	1	1	2	3	5	8	13	21	34	55
F_{-n}	0	1	-1	2	-3	5	-8	13	-21	34	-55
L_n	2	1	3	4	7	11	18	29	47	76	123
L_{-n}	2	-1	3	-4	7	-11	18	-29	47	-76	123

The terms of the F_n - and L_n - sequences have some wonderful mathematical properties. For example, for the odd $n = 2k+1$ the terms of the F_n and F_{-n} sequences coincide, i.e. $F_{2k+1} = F_{-2k-1}$ and for the even $n = 2k$ they are opposite by sign, that is, $F_{2k} = -F_{-2k}$. As for Lucas numbers L_n, it is the contrary, i.e. $L_{2k} = L_{-2k}$; $L_{2k+1} = -L_{-2k-1}$.

It is easy to determine that L_n and F_n are connected each to other by the following relations:

$$L_n = F_{n-1} + F_{n+1}; \qquad L_n = F_n + 2F_{n-1}; \qquad L_n + F_n = 2F_{n+1}.$$

But who is the author of Lucas numbers? In the 19th century the interest in Fibonacci numbers and golden section in mathematics increases. The scientific works of the French mathematician Lucas are especially noticeable in this respect. In [4] we can find the brief Lucas biography.

"François-Édouard-Anatole Lucas (4.4.1842 – 8.10.1891) is the French mathematician, professor. He was born in Amjen. He worked in the lyceum of Lunle-Gran in Paris. The major works of Lucas fall into number theory and indeterminate analysis. In 1878 Lucas gave the criterion for definition of the primarity of Mersenn's numbers of the kind $M_p = 2^p - 1$*. Applying his method Lucas established, that the number of* $M_{127} = 2^{127} - 1$ *is the prime one. During 75 years this number was the greatest prime number known for science. Also he found the 12th perfect number and formulated a number of interesting mathematical problems. Lucas believed that with the help of machines or other devises the addition is more convenient to perform in the binary number system, than in the decimal one".*

Let us give some explanations to Lucas' scientific outcomes. It is well known that the prime numbers are called such numbers, which have not other divisors except for themselves and the unit of 1, namely: 2, 3, 5, 7, 11, 13, …. Still Pythagoreans proved that a number of the prime numbers is infinite (the proof of this statement is contained in the "Euclidean Elements"). The analysis of the prime numbers and finding out of their distribution in natural number series is rather difficult problem of number theory. Therefore scientific outcome obtained by Lucas in the field of the prime numbers, doubtlessly, belonged to category of outstanding mathematical achievements.

From the historical point of view it is interesting that Lucas already in the 19th century, that is, long before originating modern computers, paid attention on technical advantage of the binary number system, that is, he almost for one century anticipated "John von Neumann Principles" underlying modern electronic computers.

But for our Museum most relevant is the fact that that just Lucas attracted attention to remarkable numeric sequence 1, 1, 2, 3, 5, 8, 13, 21, 34, …, which was called by him *Fibonacci numbers* in honor of the author of this sequence, Leonardo Pisano Fibonacci, who introduced them in the 13th century.

Also Lucas introduced a concept of the generalized Fibonacci numbers, which are computed according to the following recurrent formula:

$$G_n = G_{n-1} + G_{n-2}, \qquad (11)$$

but for the different initial terms G_1 and G_2. For example, the sequence of numbers 3, 8, 11, 19, 30, 49, … falls into the class of the generalized Fibonacci numbers.

After Lucas the mathematical works on Fibonacci numbers, according to saying of one mathematician, *"begun to propagate as Fibonacci's rabbits"* - and just in that there is a historical Lucas' contribution to Fibonacci number theory!

2.16. Binet's formulas. First mathematical connection between the golden ratio and Fibonacci & Lucas numbers was established in the 19th century by the well-known French mathematician Binet.

We have the following information about the French mathematician Jacques Philippe Marie Binet, the 19th century enthusiast of Fibonacci numbers.

Jacques Philippe Marie Binet (1776-1856)

He was born on February 2, 1776 in Renje and died on May 12, 1856 in Paris. Binet graduated from the Polytechnic School in Paris and after its graduation in 1806 he worked at the Bridges and Roads Department of the French government. He became a teacher of the Polytechnic school in 1807 and in one year became assistant-professor of the applied analysis and descriptive geometry. Binet studied foundation of matrix theory and his works in this field were continued then by other researchers. He discovered in 1812 the rule of matrix multiplication and already this discovery glorified his name more than other his works. Except for mathematics Binet worked and in other areas. He published many articles on mechanics, mathematics and astronomy. In mathematics Binet introduced the notion of the "beta function"; also he considered the linear difference equations with alternating coefficients and established some metric properties of conjugate diameters and so on. Among different honors obtained by Binet even at his life it is necessary to mention that he was selected to the Parisian Academy of Sciences in 1843.

But the following fact is the most interesting for our Museum. Binet studied the linear recursive equations whose partial case is Fibonacci recursive formula (7). Apparently, just this fascination resulted him in the famous Binet's formulas connected Fibonacci and Lucas numbers to the golden ratio. Let us remind that Binet's formulas in mathematics are well known as the following group of the formulas:

$$L_n = \begin{cases} \tau^n + \tau^{-n} & with \quad n = 2k; \\ \tau^n - \tau^{-n} & with \quad n = 2k+1 \end{cases} \tag{12}$$

$$F_n = \begin{cases} \dfrac{\tau^n + \tau^{-n}}{\sqrt{5}} & with \quad n = 2k+1; \\ \dfrac{\tau^n - \tau^{-n}}{\sqrt{5}} & with \quad n = 2k \end{cases} \tag{13}$$

where L_n и F_n are Lucas and Fibonacci numbers respectively, $\tau = \frac{1+\sqrt{5}}{2}$ is the golden ratio.

What mean the formulas (12), (13)? The formula (12) means that the *n*-th Lucas number L_n can be presented or as the sum of the golden proportion degrees $\tau^n + \tau^{-n}$ for the even values of

$n=2k$ or as their difference $\tau^n - \tau^{-n}$ if $n=2k+1$. The formula (13) asserts that for representation of the n-th Fibonacci number F_n it is necessary to make the same, that is, to compute the sum $\tau^n + \tau^{-n}$ for the odd values of $n=2k+1$ or their difference $\tau^n - \tau^{-n}$ for the case $n=2k$ and then to divide this sum or difference by the irrational number of $\sqrt{5}$.

It is necessary to note that Binet's formulas (12), (13) can be attributed to a class of the outstanding mathematical formulas joined two wonderful mathematical notions introduced in the ancient mathematics, namely, *integer numbers* and *irrational numbers.* Really, the left-hand parts of the formulas (12), (13) are always Lucas or Fibonacci numbers, that is, integer numbers, while the right-hand parts of the formulas (12), (13) are always some combinations of the irrational numbers τ^n, τ^{-n} and $\sqrt{5}$. For example, it is impossible to imagine that Fibonacci number $F_7 = 13$ can be represented as the following:

$$F_7 = 13 = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^7 + \left(\frac{1+\sqrt{5}}{2}\right)^{-7}}{\sqrt{5}}$$

but this surprising formula is only a partial case of the general formula (13).

Also note that Lucas' and Binet's researches in Fibonacci field became by the launch pad for the group of the American 20th century mathematicians organized in 1963 the Fibonacci Association and begun to issue "The Fibonacci Quarterly" since 1963.

2.17. The regular icosahedron as the main geometric figure of mathematics. Among the five "Platonic Solids" the regular icosahedron and dodecahedron (Fig. 8) take a special place. In Plato's cosmology the regular icosahedron symbolizes water, and dodecahedron does harmony of the Universe. These two "Platonic Solids" are connected directly to "pentagram" and through it to the golden ratio. Dodecahedron and regular icosahedron form the basis of so-called *"icosaedral-dodecahedral doctrine"* running through all history of human culture, starting since Pythagoras, Plato, Euclid, Kepler and up to now.

And probably, it is impossible to consider accidental that this doctrine got unexpected development in the works of the outstanding German mathematician Felix Klein.

Felix Klein (1849-1925)

Felix Klein was born in 1849 and died in 1925. He graduated from the University Bonn. Since 1875 he worked as a Professor of the Higher Technical School in Munich, since 1880 as a Professor of the University of Leipzig. In 1886 he moved to Gettingen where he headed the Mathematical Institute of the University of Gettingen; during the first quarter of the 20th century this Mathematical Institute was recognized as the World mathematical center. The main Klein's works were dedicated to Non-Euclidean geometry, theory of continuous groups, theory of algebraic equations, theory of elliptic functions, etc. His ideas in the field of geometry was stated by Klein in the work "Comparative consideration of new geometrical researches" (1872) known under the title "Erlanger Program".

According to Klein, every geometry is an invariant theory for a special group transformation. Dilating or narrowing down this group it is possible to pass from one type of geometry to other. The Euclidean geometry is the science about the metric group invariants, projective geometry about the projective group invariants, etc. A classification of group transformations gives us the classification of geometries.

Klein's research concerns also upon regular polyhedrons. His book "*The Lectures about a regular icosahedron and solution of the 5th degree equations"* published in 1884 is dedicated to this problem. Though the book is dedicated to the solution of the 5-th degree algebraic equations, but the main idea of the book is much deeper and is dedicated to a role of the Platonic Solids, in particular of the regular icosahedron, in development of mathematical sciences.

According to Klein, the tissue of mathematics runs up widely and freely by sheets of the different theories. But there are mathematical objects, in which some sheets converge. Their geometry binds the sheets and allows enveloping a general mathematical sense of the miscellaneous theories. The regular icosahedron, in Klein's opinion, is just similar mathematical object. *Klein treats the regular icosahedron as the mathematical object, from which the branches of the five mathematical theories appear, namely geometry, Galois' theory, group theory, invariants theory and differential equations.*

Thus, the main Klein's idea is extremely simple:

"Every unique geometrical object is connected somehow or other to properties of the regular icosahedron".

In what is a significance of Klein's ideas since the point of view of the harmony theory? First of all we can see that the regular icosahedron, one of the "Platonic Solids", is selected as the geometric object integrating the "main sheets" of mathematics. But the regular dodecahedron is based on the golden section! It follows from here that just the golden section is the main geometrical proportion, which, following to Klein, can join all branches of mathematics.

Klein's contemporaries could not understand and access properly revolutionary significance of Klein's "icosahedral" idea. Its significance was accessed properly equally in 100 years, that is, only in 1984 when the Israel scientist Dan Shechtman published the article verifying an existence of special alloys (called quasi-crystals) having so-called "icosaedral" symmetry, that is, the 5-th order symmetry, which is strictly forbidden in the classic crystallography.

Thus, still in the 19th century the ingenious Klein's intuition resulted him in the thought that one of the most ancient geometrical figures, the regular icosahedron, is the main geometrical figure of science, in particular, mathematics. Thereby Klein inhaled in the 19th century a new life in development of the "icosaedral-dodecahedral" doctrine about the Universe structure; this doctrine was developed by the great scientists and philosophers namely Plato who constructed his cosmology on the basis of the regular polyhedrons, Euclid who devoted his "Elements" to presentation of the “Platonic Solids” theory, Johannes Kepler who used the "Platonic Solids" in his rather original geometrical model of the Solar system, and many others.

3. The Golden Section, Nature, and Man

This hall consists of two exhibitions:

(1) The Golden Section in Nature, and

(2) The Golden Section and Man.

In the former exhibition, numerous applications of the golden section (pentagonal symmetry, golden spirals, and Fibonacci numbers, for example) are given. In the latter, examples of painting and sculpture are used to illustrate the golden section as a formula of a beauty.

3.1. The "golden" spirals and "pentagonal" symmetry in the alive Nature. The "golden" spirals are widespread widely in the biological world. For example, animal horns grow only from one end. This growth is realized on the equiangular spiral. It was proved that among different kinds of spirals showing in horns of rams, goats, antelopes and other horned animals the "golden" spirals meet most often.

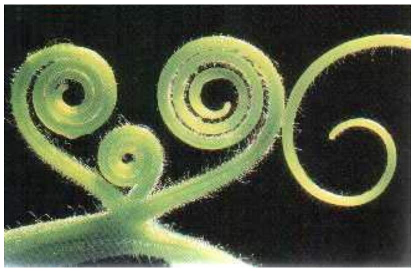


Figure 18. The “golden” spirals in animal horns and plants

The spirals widely show themselves in the alive nature. The plant tendrils are became twisted by spirals, the growth of tissues in tree's trunks is realized by spiral there, the sunflower seeds are arranged on the spirals, the helical motions are watched at growth of the roots and sprouts. Apparently, in it the heredity of planet organization shows, and it is necessary to search for its roots on the cell-like and molecular level.

A majority of shells have spiral shape.

Figure 19. The “golden” spirals in shells

Studying construction of shells, scientists paid attention to expediency of the shapes and surfaces of shells: the internal surface is smooth, the outside one is fluted. The mollusk body is inside shell and the internal surface of shells should be smooth. The outside edges of the shell augment a rigidity of shells and, thus, increase its strength. The shell forms astonish by their perfection and profitability of means spent on its creation. The spiral's idea in shells is expressed n the perfect geometrical form, in surprising beautiful, "sharpened" design.

For some mollusks a number of parts reshaping conical shells corresponds to Fibonacci numbers. So, the shell of foraminifer has 13 parts, a number of chambers of the shell of nautilus is equal to 34, the shell of gigantic tridacne is collected in 5 folds. These and many others examples show that shell constructions of many fossil and modern mollusks prefer Fibonacci numbers: 5, 8, 13, 21, 34.

In the alive nature the forms based on the "pentagonal" symmetry (marine asters, marine hedgehogs, flowers) are widespread widely. The flowers of water lily, wild rose, hawthorn, small nail, pear, apple, strawberry and many other flowers are five-petal. The flower of the Chinese rose with the brightly expressed "pentagonal" symmetry is shown below.

a)

b)

c)

d)

Figure 1.20. Examples of the “pentagonal” symmetry in Nature:
(a) Chine rose; (b) Apple in cut; (c) Marine Star; (d) Cactus

However the marine asters own not only the "pentagonal" symmetry. In the Pacific Ocean there are the marine asters with 8 and 13 rays. The marine aster "sunflower" has 33 rays, and "fiery" aster has 55 rays. Thus, for many marine asters a number of rays corresponds to Fibonacci numbers or close to them numbers.

An availability of five fingers on human hand or bone's embryos on the organs of a man and many animals are additional testimony of the pentagonal form and the golden section spreading in morphology of the biological world.

3.2. Omnipresent phyllotaxis. All in the Nature is subordinated to the stringent mathematical laws. It appears, that the arrangement of leafs on stems of plants also has a stringent mathematical nature and this phenomenon is called "phyllotaxis" in botanic. An essence of phyllotaxis consists in screw arrangement of leafs on plant stems (branches on trees, petals in racemes etc.). In the phyllotaxis phenomenon the more complicated concepts of symmetry, in particular, the concept of the "screw axis of symmetry", are used. Let us consider, for example, arrangement of leafs on the plant stem (Fig. 21). We can see that leafs are at different altitudes of the stem along the screw curve winded around of its stem. To pass from the underlying leaf to the next one it is necessary mentally to turn the leaf on some angle around of the vertical axis and then to raise it on a definite distance up. In it there exists an essence of "screw" symmetry.

Figure 21. A "screw" symmetry.

And now let us consider characteristic "screw axes" arisen on plant stems (Fig.22). In Fig.22-à the stem of plant with the symmetry screw axis of the third order is shown. Let us observe the line of leaf-arrangement in this figure. To pass from the leaf of 1 to the leaf of 2, it is necessary to turn the leaf of 1 around of the stem axis on 120° counter-clockwise (if to look from below) and then to move the leaf of 1 along the stem in vertical direction so long as it will be combined with the leaf of 2. Repeating similar operation we can pass from the leaf of 2 to the leaf of 3 and so on. It is necessary to attract attention to the fact that the leaf of 4 lies above of the leaf of 1 (as though repeats it, but its level is higher). Note that moving from the leaf of 1 to the leaf of 4 we made turn triply on the angle 120°, i.e. we executed the full revolution around of the stem axis (120° × 3 = 360°).

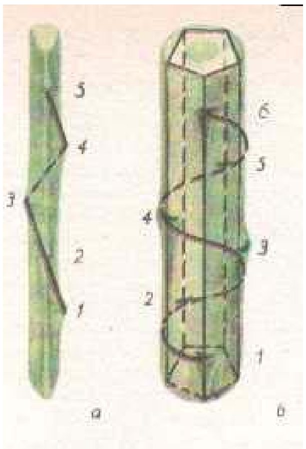


Figure 22. The screw stem axis of symmetry.

Botanists are called the turn angle of the screw axis as the "leaf divergence angle". The vertical straight line connecting two leafs arranged one the stem one above another is named the *"ortho-line"*. The line segment 1-4 of the "ortho-line" corresponds to the full translation of the screw axis. As we will see further a number of the revolutions around of the stem axis for transition from the lower leaf to the upper one arranged exactly above lower (on the "ortho-line") can be equal not only 1, but also 2, 3 and so on. This number of the revolutions is called the *"leaf cycle"*. In botanic it is custom to characterize the screw leaf-arrangement with the help of some fraction; the numerator of the fraction is equal to the "leaf cycle" and the denominator to a number of leafs in this "leaf cycle". In the case considered above we have the screw axis of the kind of 1/3.

Fig.22-b demonstrates the "pentagonal" symmetry screw axis with the "leaf cycle" of 2 (for transition from the leaf of 1 to the leaf of 6 it is necessary to make two full revolutions). The fraction describing the given axis is expressed by 2/5; the leaf divergence angle is equal to 144° (360° : 5 = 72°; 72° × 2 = 144°). Note that there are also more intricate axes, for example, of the kind of 3/8, 5/13 etc.

There is a question: what can be numbers *a* and *b* describing the screw axis of the kind of *a*/*b*? And here a Nature presents us the next surprise by the way of the so-called *"Law of phyllotaxis"*. Botanists assert that the fractions describing the plant screw axes form the stringent mathematical sequence consisting of the adjacent Fibonacci numbers ratios, that is:

$$1/2, 1/3, 2/5, 3/8, 5/13, 8/21, 13/34, \ldots . \qquad (14)$$

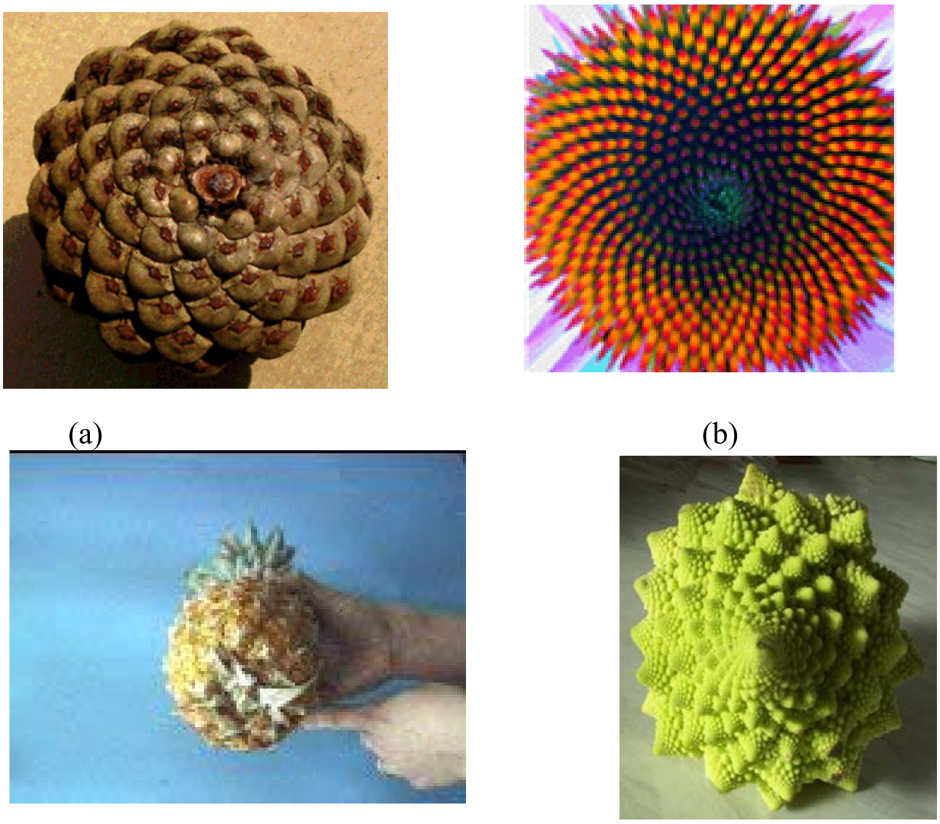
Let us remind that the Fibonacci series is the following number sequence:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots . \qquad (15)$$

Comparing (14) and (15) it is easy to see that the fractions in the sequence of (14) will be derivate by the Fibonacci numbers taken through one number.

Botanists established that the phyllotaxis fraction from the sequence of (14) are characteristic for different plants. For example, the fraction of 1/2 is peculiar to cereals, birch, grapes; 1/3 to sedge, tulip, alder; 2/5 to pear, currants, plum; 3/8 to cabbage, radish, flax; 5/13 to spruce, jasmine etc.

What is the "physical" cause underlying the "Phyllotaxis Law"? The answer is very simple. It appears that just at such arrangement of leafs on the plant stem the maximum of the solar energy inflow to the plant is reached. Taking into consideration this remark you will be not surprised also with the fact that practically all racemes and densely packaged botanic structures (pine and cedar cones, pineapples, cactuses, heads of sunflowers and many others) also strictly follow to Fibonacci numbers regularity.



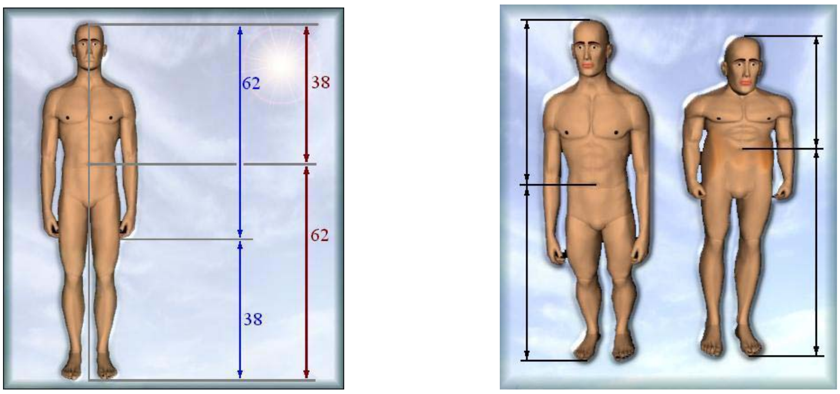
(a) (b) (c) (d)

Figure 23. Phyllotaxis structures: (a) a pine cone; (б) a head of sunflower; (в) a pineapple; (г) a head of cauliflower

3.3. The Golden Section and a Man. A human body and all its parts are subordinated to the principle of the golden proportion. Let us consider some examples.

Proportion of the human body

It is well known that a harmonic human body is divided by the navel into the golden section (Fig. 24-a). Human bodies having proportions distinguished from the golden proportion (Fig.24-b) look formlessly.



(a) (b)

Figure 24. Harmony (a) and disharmony (b) of human body

The human hand

It is proved that our hand creates a golden section in relation to your arm, as the ratio of our forearm to our hand is also 1.618, the golden ratio.

Figure 25. The human hand

Let us consider now a skeleton of our index finger (Fig.26). We can see that each section of our index finger, from the tip to the base of the wrist, is larger than the preceding one by about the golden ratio of 1.618, also fitting the Fibonacci numbers 2, 3, 5 and 8.

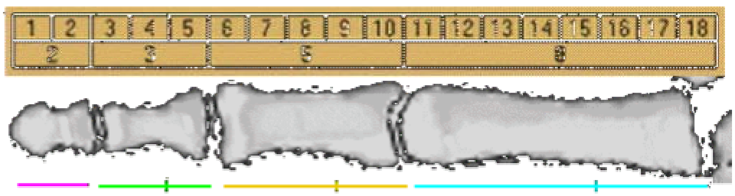


Figure 26. Human index finger

The human face

It is proved that the human face is based entirely on the golden ratio. In particular, the head forms a golden rectangle with the eyes at its midpoint. The mouth and nose are each placed at golden sections of the distance between the eyes and the bottom of the chin.

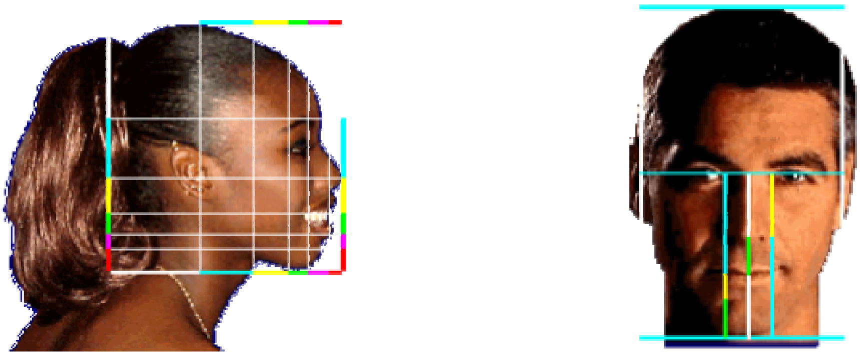


Figure 27. Proportions of human face

Note that Figures 25, 26, 27 are taken from the WEB site by Gary Meisner [8] being one of the best sites on the golden section. Analyzing Fig. 27 Gary Meisner wrote:

"The golden proportion defines the dimensions of the human profile. The blue line defines a perfect square of the pupils and outside corners of the mouth. The golden section of these four blue lines defines the nose, the tip of the nose, the inside of the nostrils, the two rises of the upper lip and the inner points of the ear. The blue line also defines the distance from the upper lip to the bottom of the chin. The yellow line, a golden section of the blue line, defines the width of the nose, the distance between the eyes and eye brows and the distance from the pupils to the tip of the nose. The green line, a golden section of the yellow line defines the width of the eye, the distance at the pupil from the eye lash to the eye brow and the distance between the nostrils. he magenta line, a golden section of the green line, defines the distance from the upper lip to the bottom of the nose and several dimensions of the eye".

3.4. Human heartbeat. Human heart is beaten uniformly (about 60 impacts in one minute in the rest state). The heart as the cylinder piston compresses and then pushes out the blood and drives it on the body. The blood pressure changes during the cardiac performance. It reaches of the greatest value in the left heart ventricle at the moment of its compression (systole). In the arteries during the heart ventricular systole the blood pressure reaches the maximum value equalling to 115-125 mm of the mercury column. At the moment of the cardiac muscle debilitation (diastole) the pressure decreases until 70-80 mm of the mercury column. The ratio of the maximum (systolic) pressure to the minimum (diastolic) pressure is equal, on the average, to 1.6, that is, it is very close to the golden proportion. Whether is this coincidence random or it reflects some objective regularity of the cardiac activity harmonic organization?

The heart is beaten continuously from man's birth up to his died. And its activity should be optimal and be subordinated to the self-organization laws of biological systems. And as the golden proportion is one of criteria of self-organizing systems naturally one may suspect that the cardiac performance is subordinated to the golden section law. One may judge about the heart activity by using the electrocardiogram, the curve reflecting different cycles of the cardiac performance.

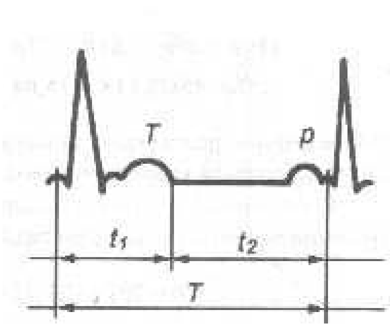


Figure 28. The human cardiogram

One may select on the cardiogram two segments of different duration corresponding to systolic (t_1) and diastolic (t_2) heart activity. It is proved [9] there exists the optimal ("golden") palpitation frequency for a man and for other mammals; here the durations of systole, diastole and full cardiac cycle (T) are in the golden proportion, that is, $T : t_2 = t_2 : t_1$. So, for example, for men the "golden" frequency is equal to 63 heart impacts in one minute, for dogs - 94 that corresponds to actual palpitation frequency in the rest state.

4. The Golden Section in Art

This hall includes the brightest examples of the golden section in music and visual art. Let us consider the most interesting of them.

4.1. The proportional scheme of the Golden Section in architecture. The book "Proportionality in the Architecture" published by the Russian architect Prof. Grimm in 1935 is well known in theory of architecture.

Prof. Grimm's book "Proportionality in Architecture" (1935)

The purpose of the book is formulated in the "Introduction" as the following:

"In view of an exceptional significance of the golden section as such proportional division, which establishes a continuous connection between the whole and its parts, and gives the constant ratio between them, which cannot be achieved by any other division, the scheme based on it advances to the first place and is adopted by us hereinafter as at check of the proportionality of historical monuments and modern facilities ... Taking into consideration this general significance of the golden section in all developments of architectural thought, it is necessary to recognize the proportionality theory based on the division of the whole into proportional parts adequate to the terms of the "golden" geometrical progression as the basis of architectural proportionality in general".

Prof. Grimm considers the golden section of the line segment *AB* by the point *C* into two unequal parts and names the large part *CA* the *major*, and the smaller part *AC* the *minor*. Behind Luca Pacioli after careful exploration of the golden section Grimm establishes a number of "exceptional" geometric properties of the golden section and made the following conclusion:

"In general it is necessary to recognize the extremely outstanding property of the golden section, which cannot be reached by arithmetic mean proportions, especially by other divisions of the whole".

Grimm confirms his idealized surveys in the field of the "golden" proportional scheme by the architectural examples from the art of classics (Parthenon, Jupiter's temple in Tunis), monuments of the Byzantium art, the Italian Renaissance (Sun Pietro in Montorio in Rome, Calleoni monument, Sun Peter's cathedral in Rome).

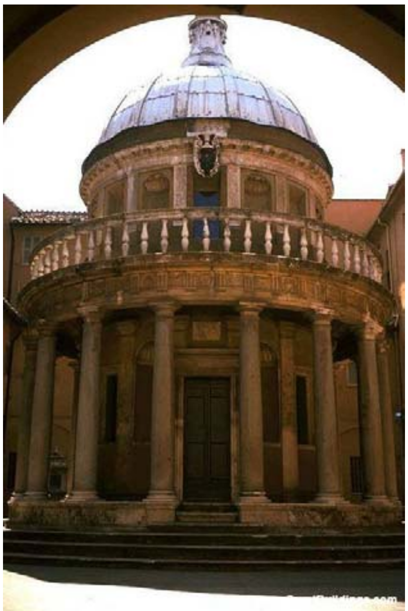


Figure 29. Sun Pietro in Montorio in Rome (Bramante).

On the first view the architecture of Baroque essentially differs from the architecture of the Classics and the Italian Renaissance and it would be possible to expect an absence of the golden section in these monuments. By analyzing of the Smolny cathedral in St.-Petersburg, which is one of the conventional monuments of this style, Grimm concludes *"that an isolation from the general scheme of the golden section in its proportions is not observed ... It is impossible to see of any conscientiously brought dissonances of proportionality, except of the well known withdrawal from the norms of classics; in any case it is indisputable and an availability of the golden section in partitioning of the basic masses of the cathedral"*.

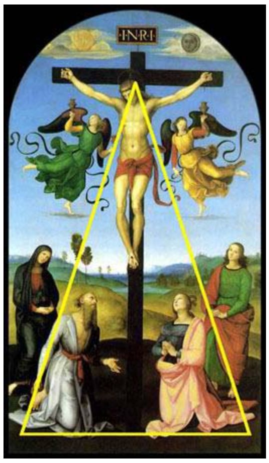
Figure 30. Smolny cathedral in St.-Petersburg (Russia)

Proportional achievements of the Russian architects, in Grimm's opinion, are based on their intuition and on their architectural-art searches. Nevertheless, in the best monuments we meet repeated application of the golden ratio. As an example of such architectural monument Grimm considers the campanile of the Christmas Christly church in Yaroslavl, in which *"as well as in other old Russian monuments, a rather essential coordination with the golden section in the main their masses is seen"*.

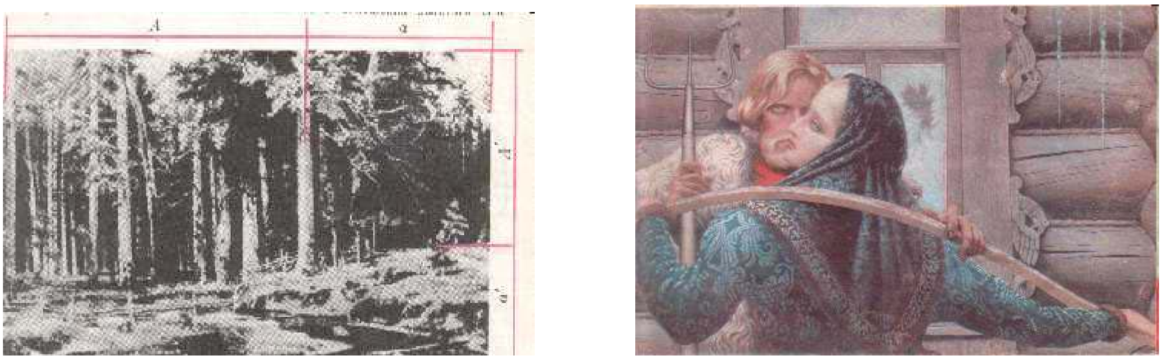
4.2. Chopin's etudes in lighting of the Golden Section. Any musical composition has a temporary duration and is divided into separate parts by some "aesthetic stakes", which facilitate our perception of the piece. The Russian musicologist Sabaneev in his article [10] shows that the separate time intervals of the musical pieces connected by the "culmination event" are, as a rule, in the ratio of the golden section. In Sabaneev's opinion, the quantity and frequency of the golden section usage in musical composition depends on the genius of the composer. Those musical pieces distinguished by the most frequent use of the golden section come from the most brilliant composers, that is, the intuition of the form and ordering, as it is necessary to expect, is strongest for the first class of geniuses.

According to Sabaneev's observations, the greatest number of musical pieces based on the golden section are observed in works of Arensky (95%), Beethoven (97%), Gaidn (97%), Mozart (91%), Scriabin (90%), Chopin (92%), Schubert (91%). Chopin's etudes were studied by Sabaneev in detail. He found that 154 exhibited the golden section. In some cases the construction of the musical piece combined symmetry and the golden section simultaneously; in these cases the piece was divided in some symmetrical way, and the golden section was observed in each. For example, many of Beethoven's compositions are divided into two symmetrical parts, and the golden section is observed in each part.

4.3. The Golden Section in painting. Exploring the compositional structures of paintings, the masterpieces of world art, critics have observed that the proportion of the golden section is widely used in art works of Michelangelo (pentagram), Rafael Santi ("golden" triangle), Ivan Shishkin, Konstantin Vasil'ev ("golden" rectangle).



(a) *Holy Family* by Michelangelo (b) *Crucifixion* by Rafael Santi



(c) *The Ship Grove* by Ivan Shishkin (d) *Near to the window* by Konstantin Vasil'ev

Figure 31. Examples of the Golden Section application in painting

Phyllotaxis lattices

In the world of botany, Fibonacci numbers and the golden section may be seen in "phylotaxis" or growth patterns of plants. For example, cactus areoles (concentrations of thorns) are placed in spirals, and the numbers of left-hand and right-hand spirals are the consecutive Fibonacci numbers 21 and 34. Cactus' areoles can be presented on a plane as the following raster lattice having 21 lines with right-hand slope and 34 lines with left-hand slope (Fig. 32).

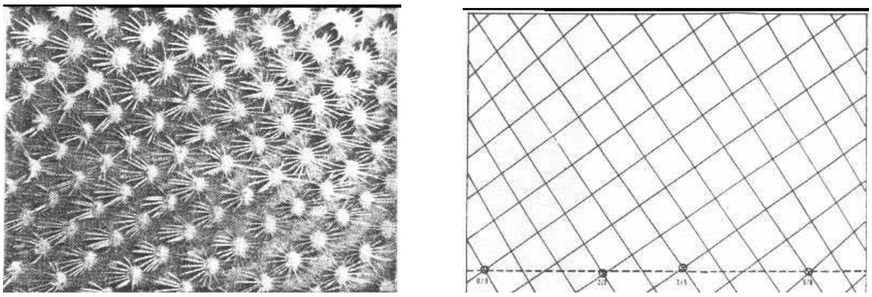


Figure 32. Cactus surface and its geometrical model

The Austrian scientist Paturi, the author of the remarkable book "The Plants as Ingenious Engineers of Nature" [11], found the use of raster lattices in the paintings of the great Renaissance artists; in particular, in Titian's painting "Vakch and Ariadna". He wrote: "*In all times the artists, consciously or unconsciously, studied to comprehend the laws of aesthetic perception by watching nature. The artists were enchanted always by the simple and simultaneously rational geometry of the biological growth forms.*"

5. Modern Fibonacci Mathematics and Computer Science

This hall of the Museum contains exhibitions very important since scientific point of view. The demands on higher standard of mathematical knowledge and is intended for scientists and mathematicians. But we try to state this material maximum popularly.

5.1. "Ardor of chilling numbers". Why just Fibonacci numbers were selected by Nature, Science and Art to be measure of harmony and beauty? Answering this question is hiding in their wonderful mathematical properties. Let us consider some of them.

In the table below the Fibonacci numbers with the odd indexes are colored by yellow and with the even indexes by blue. We can see that all the yellow numbers in the lower rows of the table coincide and all the blue numbers are opposite by sign. It is impossible to imagine that this regularity is true for all the values of the indexes n from +∞ until -∞. This mathematical fact causes feeling of rhythm and incomprehensible harmony.

n	0	1	2	3	4	5	6	7	8	9	10
F_n	0	1	1	2	3	5	8	13	21	34	55
F_{-n}	0	1	-1	2	-3	5	-8	13	-21	34	-55

Let us consider the ratios of the two adjacent Fibonacci numbers being mathematical basis of the "Phyllotaxis Low":

$$1/1,\quad 2/1,\quad 3/2,\quad 5/3,\quad 8/5,\quad 13/8,\quad 21/13,\quad 34/21,\quad 55/34, \ldots \tag{16}$$

It is well known this numerical sequence strives in limitation to the golden ratio $\tau = \frac{1+\sqrt{5}}{2}$.

However, an approach of the sequence (16) to the golden ratio has rhythmic, pulsed character because all the yellow/blue ratios in (16) always are more then τ and all the blue/yellow ratios are less then τ. Deviations of the ratios (16) from τ decrease as n increases and strive to 0. However, the ratios (16) cannot be the same as τ because the ratios (16) are rational numbers and τ is an irrational one.

Let us take from the table the Fibonacci number of 5 and then let us calculate its square, that is, $5^2 = 25$. Now we take the product of the two adjacent Fibonacci numbers 3 and 8 encircled the Fibonacci number of 5, that is, 3×8 = 24. Then we can record:

$$5^2 - 3\times 8 = 1.$$

And now we will do the same mathematical operations for the next Fibonacci number of 8, that is, at first we square it ($8^2 = 64$), after that we calculate the product of the two adjacent to 8 Fibonacci numbers of 5 and 13 (5×13 = 65) and then we subtract number of $8^2 = 64$ from the number of 65:

$$8^2 - 5\times 13 = -1.$$

Note the obtained difference is equal to (-1).

Further we have:

$$13^2 - 8\times 21 = 1;$$

$$21^2 - 13\times 34 = -1 \text{ and so on.}$$

We can see, that the square of some Fibonacci number F_n always differs from the product of the two adjacent Fibonacci numbers F_{n-1} and F_{n+1} encircled it by 1 and the sign of this 1 depends on the index n of the Fibonacci number F_n . If the index n is even then the number of 1 undertakes with minus, and if odd, with plus. The indicated property of the Fibonacci numbers can be expressed by the following mathematical formula:

$$F_n^2 - F_{n-1}F_{n+1} = (-1)^{n+1}. \tag{17}$$

This wonderful formula evokes a reverent thrill if we imagine that this fact is valid for any value of n (we remind that n can be some integer in limits since -∞ up to +∞), and gives genuine aesthetic enjoying because the alternation of + 1 and -1 in the expression of (17) at successive oversight of all the Fibonacci numbers produces no realized feeling of a rhythm and harmony.

5.2. Hyperbolic Fibonacci and Lucas functions. As is well known a number of irrational numbers is limitless. However, some of them occupy a special place in the history of mathematics, moreover in the history of material and spiritual culture. Their importance consists of the fact that they express some relations having universal character and appearing in the most unexpected applications. The π-number and Euler's number of *e* are the most important from them. The π-number expressing a ratio of the circle length to its diameter entered mathematics in the ancient period along with trigonometry, in particular spherical trigonometry considered as the applied mathematical theory intended for calculation of the planet coordinates on the «celestial spheres» («the cult of sphere»). The *e*-number entered mathematics much later than the π-number. Its discovery was immediately connected to the discovery of *Natural Logarithms*. As is well known the *e*-number expresses a number of the important geometric properties of the *hyperbola*.

The π- and *e*-numbers "generate" a variety of the fundamental functions called the "*elementary functions"*. The π-number "generates" the *trigonometric functions sin x* and *cos x* , the *e*-number "generates" the *exponential function* e^x, the *logarithmic function* $\log_e x$ and the *hyperbolic functions* namely the *hyperbolic sine* and the *hyperbolic cosine*:

$$sh\, x = \frac{e^x - e^x}{2}; \qquad ch\, x = \frac{e^x + e^x}{2}. \tag{18}$$

The trigonometric functions *sin x* and *cos x* are connected one to other with the following wonderful formula:

$$\sin^2 x + \cos^2 x = 1 \tag{19}$$

well-known for everyone studied trigonometry.

Also it is well-known that the hyperbolic functions (18) are connected one to other with the following formula:

$$ch^2 x - sh^2 x = 1 \tag{20}$$

As is well known the hyperbolic functions (18) play a fundamental role in development of mathematics and physics. When the famous Russian 19th century geometer Nikolay Lobatchevsky elaborated a new geometry, *Lobatchevsky's geometry,* he used just the hyperbolic functions (18) for simulation of geometric relations of new geometry. And when the famous German 20th century mathematician Herman Minkovsky gave geometric interpretation of Einstein's theory of relativity he introduced a four-dimensional space with hyperbolic metric based on (18).

Recently a new class of the hyperbolic functions, so-called *hyperbolic Fibonacci and Lucas functions,* was introduced into being [20]. Let us write Binet's formulas for Fibonacci numbers and Lucas numbers in the following form:

$$F_{2k} = \frac{\tau^{2k} - \tau^{-2k}}{\sqrt{5}}; \qquad F_{2k+1} = \frac{\tau^{2k+1} + \tau^{-(2k+1)}}{\sqrt{5}} \tag{21}$$

$$L_{2k+1} = \tau^{2k+1} - \tau^{-(2k+1)}; \qquad L_{2k} = \tau^{2k} + \tau^{-2k}, \tag{22}$$

where $k = 0, \pm 1, \pm 2, \pm 3, \ldots$.

Comparing the formulas of (21), (22) to the hyperbolic functions of (18) we can see surprising similarity between them. This fact was a basis for introduction of the Fibonacci and Lucas hyperbolic functions described in [20]. With this in mind let us replace the discrete variable k in the formulas of (21), (22) by the continues variable x and introduce the following definitions for Fibonacci and Lucas hyperbolic functions:

Fibonacci hyperbolic sine and cosine

$$sFx = \frac{\tau^{2x} - \tau^{-2x}}{\sqrt{5}}. \qquad cFx = \frac{\tau^{2x+1} + \tau^{-(2x+1)}}{\sqrt{5}}. \tag{23}$$

Lucas hyperbolic sine and cosine

$$sLx = \tau^{2k+1} - \tau^{-(2k+1)}. \qquad cLx = \tau^{2k} + \tau^{-2k}. \tag{24}$$

Note that for the discrete values $x = k$ the Fibonacci and Lucas hyperbolic functions are coincident with the Fibonacci and Lucas numbers because

$$sFk = F_{2k}; \quad cFx = F_{2k+1}; \quad sLx = L_{2k+1}; \quad cLx = L_{2k}. \tag{25}$$

Of what importance have new classes of the hyperbolic functions for general science and mathematics, in particular? Let us begin from the theory of Fibonacci numbers. Until now the theory of Fibonacci numbers develops as discrete theory because Fibonacci numbers are a part of natural numbers and belong to the discrete set. But the Fibonacci and Lucas hyperbolic functions are "continues" mathematical objects and we can apply methods of "continues" mathematics (in particular, differentiation and integration) to explore these functions. But every mathematical identity for the Fibonacci and Lucas hyperbolic functions has a "Fibonacci" interpretation using (25) and every identity for Fibonacci and Lucas numbers can be interpreted as some identity for the Fibonacci and Lucas hyperbolic functions.

As example let us find "hyperbolic" interpretation of the identity (17). We can write the formula (17) as a pair of the two formulas for the even (n=2k) and odd (n=2k+1) values of the discrete variable n:

$$F_{2k}^2 - F_{2k-1}F_{2k+1} = (-1)^{2k+1} = -1; \tag{26}$$

$$F_{2k+1}^2 - F_{2k}F_{2k+2} = (-1)^{2k+2} = 1. \quad (27)$$

Let us consider now the formulas (26), (27) since the hyperbolic Fibonacci functions point of view. Using (25) we can write the formulas (26), (27) as the following:

$$(sFk)^2 - cF(k-1) \times cFk = -1; \quad (28)$$

$$(cFk)^2 - sFk \times cF(k+1) = 1, \quad (29)$$

where k=0, ±1, ±2, ±3, … .

However, the formulas (28), (29) are a partial case of the following general formulas taken by us without proof:

$$(sFx)^2 - cF(x-1) \times cFx = -1. \quad (30)$$

$$(cFx)^2 - sFx \times cF(x+1) = 1. \quad (31)$$

Thus, we have found a new class of the hyperbolic functions given with (23) and (24) having remarkable properties given with (30) and (31). These functions keep all properties of the classical hyperbolic functions (18) however, their main attribute is the fact that for discrete values of x=k (k=0, ±1, ±2, ±3, …) the Fibonacci and Lucas hyperbolic functions (23) and (24) coincide with Fibonacci and Lucas numbers. Now let us imagine that Lobatchevsky and Minkovsky did know the functions (23) and (24) and use them in their geometric theories. But then they could conclude that the geometric space has Fibonacci's property. It looks like the Ukrainian architect Oleg Bodnar has realized this idea by means of development of the new phyllotaxis theory based on the Fibonacci and Lucas hyperbolic functions [24].

5.3. Fibonacci numbers in Pascal Triangle. In our daily life we use widely the mathematics branch called *combinatorial analysis*. This one studies so-called *finite sets*. The set consisting of n elements is called n-element one. However we can chose k elements from n-element set. Each k-element part of the n-element set is called *combination from given n elements by k*. One of the problems of combinatorial analysis is to find *a number of combinations of n elements by k.* Usually this number is marked as C_n^k and called *binomial coefficients.*

There exists a special method calculating binomial coefficients. This one is called *Pascal's method*; it is reduced to construction of special numerical table called *Pascal Triangle.* We can see that the top row (the 0-row) of Pascal triangle consists of 1's and all diagonal binomial coefficients are equal to 1. Each binomial coefficient inside Pascal triangle are calculated according to *Pascal's rule*:

$$C_n^k = C_{n-1}^k + C_n^{k-1}.$$

Pascal Triangle

1	1	1	1	1	1	1	1	1	1
	1	2	3	4	5	6	7	8	9
		1	3	6	10	15	21	28	36
			1	4	10	20	35	56	84
				1	5	15	35	70	126
					1	6	21	56	126
						1	7	28	84
							1	8	36
								1	9
									1
1	2	4	8	16	32	64	128	256	512

If now we sum up all the binomial coefficients of the *n*-th column we get the binary number 2^n , that is, 1, 2, 4, 8, 16, ...

Let us shift now each row of Pascal triangle in one column to the right about the preceding row. As the result of such transformation we get the following number array called the *1-Pascal triangle*:

1-Pascal triangle

1	1	1	1	1	1	1	1	1	1	1	1
		1	2	3	4	5	6	7	8	9	10
				1	3	6	10	15	21	28	36
						1	4	10	20	35	56
								1	5	15	35
										1	6
1	1	2	3	5	8	13	21	34	55	89	144

It is easy to prove that the sum of the binomial coefficients in the *n*-th column of the 1-Pascal triangle is equal to Fibonacci number F_{n+1}.

If we shift each row of the initial Pascal Triangle in the *p* columns to the right about the preceding row (p = 0, 1, 2, 3, ...), we get number array called the *p-Pascal triangle*. Summing up binomial coefficients by columns we get so-called *p-Fibonacci numbers* given by the following recursive formula:

$$F_p(n) = F_p(n-1) + F_p(n-p-1) \quad \text{with} \quad n>p+1; \tag{32}$$

$$F_p(1) = F_p(2) = ... = F_p(p+1) = 1. \tag{33}$$

Note that the recursive formula of (32), (33) generates infinite number of numerical sequences because each *p* (*p*=0, 1, 2, 3, …) generates own numerical sequence. In particular, *p*=0 generates the binary numbers: 1, 2, 4, 8, 16, …; *p*=1 generates Fibonacci numbers: 1, 1, 2, 3, 5, 8, 13, … ; *p*=2 generates 2-Fibonacci numbers: 1, 1, 1, 2, 3, 4, 6, 9, 13, 19, 26, … and so on.

5.4. Generalization of the Golden Section. Let us subdivide the line segment *AB* by the point *C* according to the following proportion:

$$\frac{CB}{AC} = \left(\frac{AB}{CB}\right)^p \tag{34}$$

where p = 0, 1, 2, 3, ...

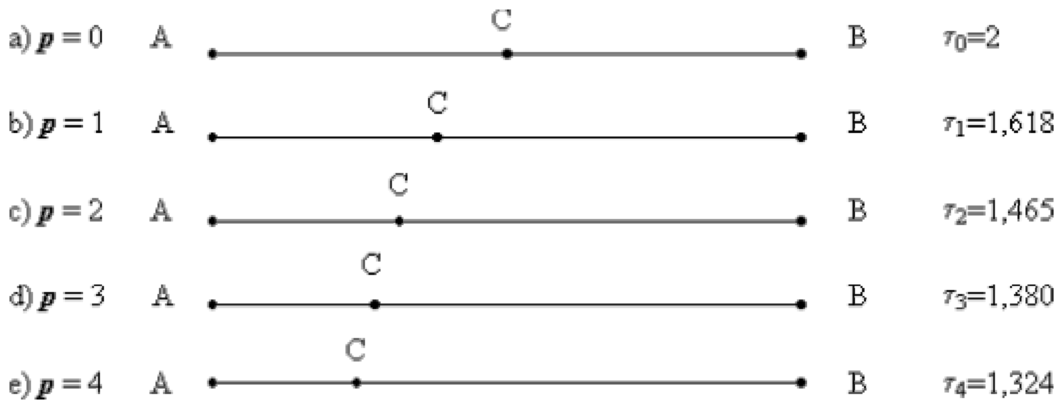


Figure 33. The Golden *p*-Sections (*p* = 0, 1, 2, 3, ...).

It is easy to prove that for the given *p* the problem (34) is reduced to the solution of the following algebraic equation:

$$x^{p+1} = x^p + 1. \tag{35}$$

Note that the proportion of (34) is reduced to the "dichotomy" for the case of $p = 0$ (Fig.33-a) and to the classical golden section for $p = 1$ (Fig. 33-b). Taking into consideration this unexpected fact the subdivision of the line segment *AB* by the point *C* in the ratio of (34) is called the *golden p-section* but the real roots τ_p of the equation of (35) are called *golden p-ratios* or *golden p-proportions*. The following property of the golden *p*-ratios emerges from the algebraic equation of (35):

$$\tau_p^n = \tau_p^{n-1} + \tau_p^{n-p-1} = \tau_p \times \tau_p^{n-1} \quad (36)$$

Note that for $p = 0$ we have $\tau_p = 2$ and the identity of (36) is reduced to the following trivial identity for binary numbers:

$$2^n = 2^{n-1} + 2^{n-1}.$$

For $p = 1$ we have $\tau_p = \tau = \frac{1+\sqrt{5}}{2}$ and the identity of (35) is reduced to the following well-known identity for the classical golden ratio:

$$\tau^n = \tau^{n-1} + \tau^{n-1} = \tau \times \tau^{n-1}$$

It is proved that the ratio of the two adjacent *p*-Fibonacci numbers $F_p(n)/F_p(n\text{-}1)$ strives to the golden *p*-ratio τ_p for the case $n \to \infty$! It means that the golden *p*-ratios τ_p form a special class of irrational numbers expressing some deep mathematical correlations of Pascal Triangle.

Thus, as result of this consideration we get a number of small mathematical discoveries:

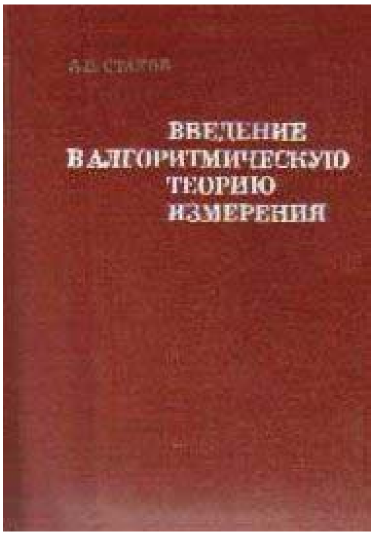
1. We have shown that there exist a fundamental connection between Pascal Triangle, Fibonacci numbers and golden ratio!
2. But by exploring Pascal Triangle we have generalized the classical Fibonacci numbers and the classical golden section and introduced the notions of the generalized Fibonacci numbers (*p*-Fibonacci numbers), the generalized golden sections (the golden *p*-sections) and the generalized golden ratios (the golden *p*-ratios). Thus, we have revealed one more secret of Pascal Triangle, which keeps in itself a new class of irrational numbers, the golden *p*-ratios (p = 0, 1, 2, 3, ...)

5.5. Applications of the *p*-Fibonacci numbers and the Golden *p*-ratios. The above introduced generalized Fibonacci numbers, the *p*-Fibonacci numbers, and generalized golden proportions, the golden *p*-proportions, became a source of new and fruitful mathematical concepts and theories. Let us consider some of them.

Algorithmic measurement theory

Fibonacci's problem of *"rabbit reproduction"* is well known and is a source of the Fibonacci numbers theory developed by the Fibonacci mathematicians. But it is less known other Fibonacci's problem called the *"problem of choosing the best system of standard weights for weighing on the balance"*. It is well known two solutions of this problem. For the former case the "optimal system of standard weights" is reduced to the binary system of weights: 1, 2, 4, 8, 16, ... It is important to note that the "optimal" measurement algorithm, arising in this case, "generates" the classic binary number system underlying modern computers. For the latter case the "optimal system of standard weights" is reduced to the "ternary" system of weights: 1, 3, 9, 27, 81, ... and this case "generates" so-called ternary symmetrical (balanced) number system used in the "ternary" computer "SETUN" designed at Moscow University in the 50th of the 20th century.

Further development of Fibonacci’s “weighing problem” in modern science was made in book [12] and brochure [13].



Staklov’s book “Introduction into Algorithmic Measurement Theory” (1977)

and Stakhov’s brochure “Algorithmic Measurement Theory” (1979)

Fibonacci codes

Fibonacci's measurement algorithms are one of the most unexpected results of the algorithmic measurement theory [12, 13]. They "generate" the following positional method of number representation:

$$N = a_nF_p(n) + a_{n-1}F_p(n - 1) + ... + a_iF_p(i) + ... + a_1F_p(1), \qquad (37)$$

where $a_i \in \{0, 1\}$ is the binary numeral of the i-th digit of the positional representation of (37); n is a number of digits in the representation of (37); $F_p(i)$ is the i-th digit weight calculated in accordance with the recurrent correlation (32), (33).

The positional representation of (37) is called *the p-Fibonacci code* [12, 13]. Note that a concept of the p-Fibonacci code includes an infinite number of the different methods of the positional number representations because every p "generates" its own p-Fibonacci code (p = 0, 1, 2, 3, ...). In particular, the case $p = 0$ generates the classical binary number system and the case p=1 generates so-called Zekendorf representation:

$$N = a_n F_n + a_{n-1} F_{n-1} + ... + a_i F_i + ... + a_1 F_1, \tag{38}$$

where F_n , F_{n-1} , …, F_1 are the classical Fibonacci numbers. The case $p = \infty$ "generates" so-called "unitary code":

$$N = 1 + 1 + \ldots + 1 \ (N \text{ times}). \tag{39}$$

Thus, the p-Fibonacci codes given by (37) is a very wide generalization of the classical binary number system and "unitary code" (39) and as if "fill out a gap" between them including them as partial cases for $p = 0$ and $p = \infty$.

Codes of the Golden p-Proportions

The irrational numbers τ_p being real roots of the algebraic equation of (35) and satisfying to the identity of (36) generate the following positional method of real numbers representation [14]:

$$A = \sum_i a_i \tau_p^i , \tag{40}$$

where $a_i \in \{0, 1\}$ and $i = 0, \pm1, \pm2, \pm3, ...$.

Stakhov's book "Codes of the Golden Proportion" (1984)

Note the formula of (40) generates an infinite number of different positional representations of real number A. In particular, the case $p = 0$ generates the classical binary representation of A, but the case $p = 1$ generates the number system with an irrational base introduced by the American mathematician George Bergman in 1957 [15]. Note that a general theory of the number systems with irrational bases τ_p is stated in the book [14].

The article [16] is further development of Begman's number system [15]. The ternary mirror-symmetrical arithmetic stated in [16] gives interesting perspectives for design of fault-tolerant processors and computers.

Fibonacci matrices and new coding theory

Let us consider so-called Fibonacci matrix Q_p introduced in [17]:

$$Q_p = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \tag{41}$$

where the index of p takes the following values: 0, 1, 2, 3, ...

Note that the Q_p-matrix is the square $(p+1)\times(p+1)$-matrix. It consists of the $p\times p$ unit matrix bordered by the last row of 0's and the first column, which consists of 0's bordered by 1's. For p = 0, 1, 2, 3, 4 the Q_p-matrices have the following forms, respectively:

$$Q_0 = (1); \quad Q_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = Q; \quad Q_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix};$$

$$Q_3 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}; \quad Q_4 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that the Q_p-matrix is a wide generalization of so-called Fibonacci Q-matrix (the case p=1) being a subject of a great enthusiasm of the Fibonacci mathematicians [6, 18] during the last decades.

It is proved [17] that

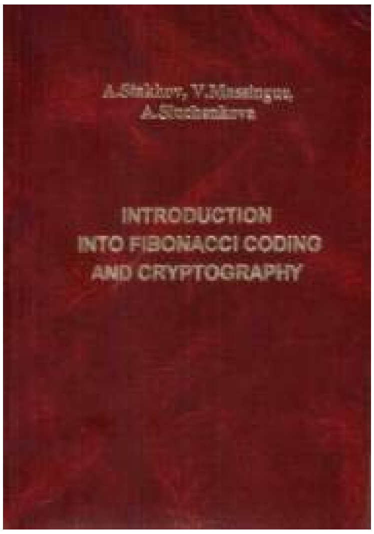
$$Q_p^n = \begin{pmatrix} F_p(n+1) & F_p(n) & \cdots & F_p(n-p+2) & F_p(n-p+1) \\ F_p(n-p+1) & F_p(n-p) & \cdots & F_p(n-2p+2) & F_p(n-2p+1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ F_p(n-1) & F_p(n-2) & \cdots & F_p(n-p) & F_p(n-p-1) \\ F_p(n) & F_p(n-1) & \cdots & F_p(n-p+1) & F_p(n-p) \end{pmatrix}. \quad (42)$$

Thus, the matrix Q_p^n is expressed through p-Fibonacci numbers $F_p(n)$ resulting from Pascal Triangle! And the result (42) is a new secret of Pascal Triangle!

Also it is proved [17] that the determinant of the matrix (26) is calculated according to the following simple formula:

$$\text{Det } Q_p^n = (-1)^{pn}. \quad (43)$$

The book [19] is devoted to statement of a new coding theory based on Fibonacci matrices (42).



The book "Introduction into Fibonacci Coding and Cryptography"

by A. Stakhov, V. Massingue and A. Sluchenkova (1999)

The essence of new coding theory based on Fibonacci matrices (42) consists of realization of the following transformation over the initial message *M* presented in matrix form:

Coding	Decoding
$M \times Q_p^n = E$	$E \times Q_p^{-n} = M$

It is clear that for the given p (p=0, 1, 2, 3, …) Fibonacci coding consists of multiplication of the initial message represented in the form of the $(p+1)\times(p+1)$-matrix *M* by the coding matrix Q_p^n given with (42). The code matrix *E* is a result of such matrix multiplication. Then the Fibonacci decoding consists of multiplication of the code matrix *E* by the "inverse" matrix Q_p^{-n}. It is proved that the determinant Det *E* of the code matrix *E* is connected to the determinant Det *M* of the initial matrix *M* with the following relation:

$$\text{Det } E = \text{Det } M \times (-1)^{pn}. \tag{44}$$

In addition, it is found that row elements of the code matrix *E* are connected between themselves by relations of the golden *p*-proportions τ_p, in particular, of the classical golden proportion $\tau = \frac{1+\sqrt{5}}{2} \approx 1,618$ for the case *p*=1. It gives for new coding method unique possibilities for detection and correction of errors in the code matrix *E.*

Harmony mathematics

At all stages of its historical development a mankind clashes with a huge number of different "worlds" surrounding it: the "world" of astronomy and mechanical movements, the "world" of electromagnetic phenomena, the "world" of random phenomena, the animal and plant "world", the "world" of Music and Art, the spiritual "world" of a Man, the social "world", the "world" of economics and business etc. For simulation of each of these "worlds" mathematics always created a corresponding mathematical discipline adequate to the studied phenomena. For example, the development of *"Newtonian Mechanics"* resulted in creation of the new mathematical discipline, *differential and integral calculus* (Newton and Leibnitz); for studying electromagnetic phenomena the *electromagnetism theory* (*"Maxwell equations"*) was created; a discovery of *"Gauss law"* became the main achievement of probability theory built for analysis of random phenomena, - and these examples could be continued.

A huge interest of modern science in Fibonacci numbers and golden section allows advancing a hypothesis about existence of the one more "world" surrounding us, the *"Fibonacc's World".* The animal and plant "world", the "world" of a Man, including his morphological, biological structure and spiritual contents, and also the "world" of Music and Art, most likely, fall into "*Fibonacci's World".* In this connection there arise an idea to create a new mathematics, *Mathematics of Harmony,* based on the golden section, Fibonacci numbers and their generalizations, the golden *p*-sections and *p*-Fibonacci numbers [21, 22].

6. Fibonaccization of modern science

This hall of the Museum is devoted to applications of Fibonacci numbers and the golden section to different areas of modern science. Let us consider the most important of them.

6.1. Quasi-crystals. A discovery of *quasi-crystals* made in 1984 by the Israeli physicist Dan Shechtman belongs to a category of revolutionary discoveries in modern physics, in particular, in crystallography. As is well known, according to the main crystallography law only the symmetry axis's of the first, second, third, fourth and sixth orders are possible for the crystals The main crystallography law rejects a possibility of the symmetry axis of the fifth order in the crystallographic lattices. The above-considered facts were canons of the traditional crystallography before Shechtman's discovery.

The alloy of the aluminum and the manganese discovered by Shechtman is formed at the super-fast cooling of the melt with the speed 10^6 K per second. Thus there is formed the alloy ordered in the pattern, which is characteristic for the symmetry of the regular icosahedron having alongside with the dodecahedron the symmetry axes of the 5^{th} order. For theoretical explanation of Shechtman's discovery researchers paid their attention to so-called *"Penrose's tiles"*. The English mathematician Penrose was engaged in the "parquet's problem" consisting of the dense filling of the plane with the help of polygons. In 1972 he found the method to cover flatness only with two simple polygons arranged non-periodically. In their simplest form "Penrose's tiles" represent a nonrandom set of the diamond-shaped figures of two types; one of them is with the interior angle of 36°, the other one with the interior angle of 72° (Fig. 34).

(a)

(b)

(c)

(d)

Figure 34. Penrose's tiles

"Penrose's tile" in Fig. 34-c can be formed by using the "golden" rhombuses in Fig. 34-a and 34-b. Fig. 36-c demonstrates the beginning of Penrose's tile construction. Let us take the 5 "golden" rhombuses of the kind of (b) and then form from them the pentagonal star. After that we add to the pentagonal star the 5 "golden" rhombuses of the kind of (a). In outcome we get a decagon in Fig. 34-c. Continuing this process, that is, adding to the decagon new "golden" rhombuses we can cover a plane by using only the two "golden" rhombuses of the kinds (a) and (b). At that there arises some non-periodic structure called "Penrose's tile". It was proved, that the ratio of the number of the "thick" rhombuses (a) to the number of the "thin" rhombuses (b) in such structure strives in limit to the golden proportion!

The parquet (mosaic) can be good clone of the quasi-crystal. The elementary space cells having a form of the regular icosahedron fill the three-dimensional space in quasi-crystal like the golden rhombuses of the kinds (a) and (b) fill two-dimensional plane in Penrose's tiles.

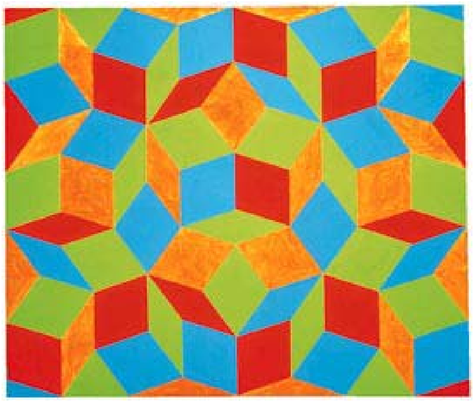
As the French scientist Gratia writes in his article "Quasi-crystals" (1988) *"the concept of the quasi-crystal presents a fundamental interest because it extends and completes the definition of the crystal. The theory based on this concept replaces the traditional idea about the "structural unit repeated in the space by the strictly periodic mode" by the key concept of the distant order. This concept resulted in widening the crystallography and we only begin to study newly uncovered wealth's. Its significance in the mineral world can be put in one row with attachment of the irrational numbers concept to the rational ones in mathematics"*.

What is a significance of the quasi-crystals discovery since our Museum point of view? First of all, this discovery is the brilliant instance of the great celebration of the "icosahedral-dodecahedral doctrine", which penetrates through all history of natural sciences and is the source of steep and useful scientific ideas. Secondly, the quasi-crystals shattered the conventional presentation about the insuperable watershed between the mineral world where the "pentagonal" symmetry was prohibited, and the alive world, where the "pentagonal" symmetry is one of most widespread.

And it is one more historical remark in conclusion. The quasi-crystal discovery made in 1984 is a worthy gift to the hundredth anniversary of the issuance of the book "Lectures on the Icosahedron .." (1884) by the famous German geometer Felix Klein who made exactly 100 years ago before the quasi-crystals discovery the brilliant prediction as to the role played by the icosahedron in future science.

The World of Matjuska Teja Krasek

The idea of Shechtman's quasi-crystals and Penrose's tiles inspired the Slovenian artist *Matjuska Teja Krasek* and she presented at the Internet (http://mitpress2.mit.edu/e-journals/Leonardo/gallery/gallery331/krasek.html) a number of interesting pictures (see below).



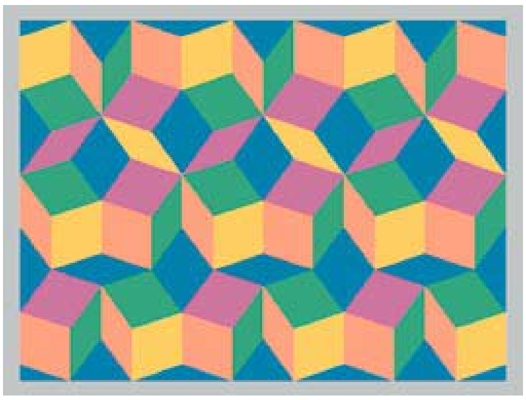
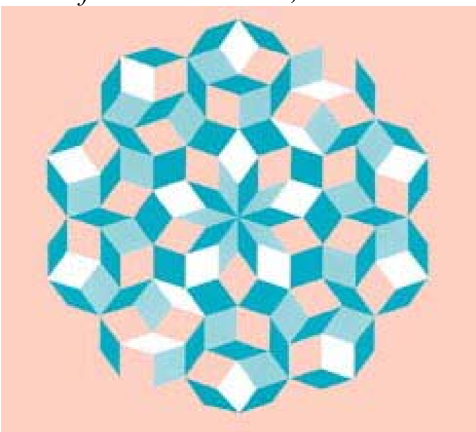


Figure 35. Quasi-crystal World (1996)	Figure 36. 9 Stars (1998)

She wrote:
"*A proportion that gives us an experience of the aesthetic appeal is the golden mean. I believe that one of the reasons we perceive it as beautiful may lay in the fact that it appears in the proportions of our own bodies, and that we can find it in the world that surrounds us".*



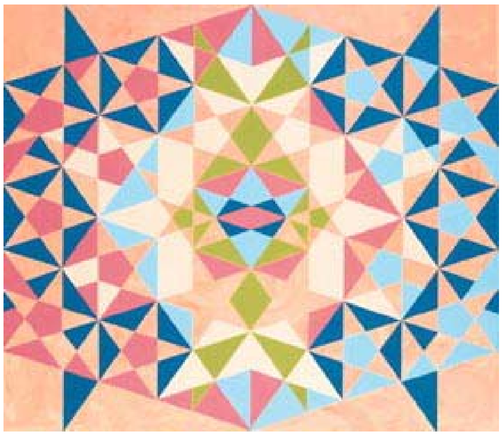


Figure 37. 10/5 (1998)	Figure 38. Quasi-cube V (1997)

She wrote:
"*I am interested in how the golden mean appears in Penrose tilings and in Penrose rhombs, which are derived from a regular pentagon where the golden mean and other interesting properties can be observed".*

6.2. Fullerenes. And now we will tell about *fullerenes,* the outstanding modern discovery in chemistry. This discovery was made in 1985, that is, several years later of the quasi-crystals discovery. The title of "fullerene" originates from *Buckminster Fuller*, (1895 –1983), who was an American visionary, designer, architect, poet, author, and inventor. Throughout his life, Fuller was concerned with the question of whether humanity has a chance to survive lastingly and successfully on the planet Earth, and if so, how. Considering himself an average individual without special monetary means or academic degree, he chose to devote his life to this question, trying to find out what an individual like him could do to improve humanity's condition that large organizations, governments, or private enterprises inherently could not do.

Fuller created a large number of inventions, mostly in the fields of design and architecture, the best-known of which is the *geodesic dome*. A geodesic dome is an almost spherical structure based on a network of struts arranged on great circles (geodesics) lying approximately on the surface of a sphere. Of all known structures made of linear elements, a geodesic dome has the highest ratio of enclosed volume to weight. Geodesic domes are far stronger as complete units than the individual constructions. A geodesic dome based on the trucuted regular icosahedron (Fig. 39) is one of Fuller's inventions.

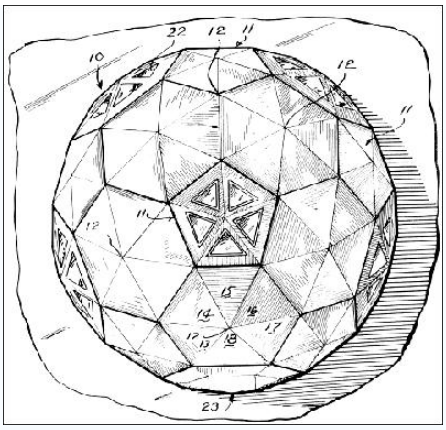


Figure 39. Fuller’s geodesic dome based on the trucuted regular icosahedron

The geodesic dome, *the Montreal Biosphere*, designed by Buckminster Fuller for the American Pavilion of Exspo 67 (Fig. 40), is Fuller’s best architectural construction.

Figure 40. The Montreal Biosphère, formely the American Pavilion of Expo 67

Fuller was the author of many inventions in design and architecture. In the U.S. postage stamp (Fig. 41) commemorating Buckminster Fuller and his contributions to architecture and science, some of his inventions are presented. Most notably, his head is shaped after one of his geodesic domes.

Figure 41. The U.S. postage stamp commemorating Buckminster Fuller

However, the American architect and scientist Buckminster Fuller became world famous after the discovery of *fullerenes.* The title of "fullerenes" refers to the closed molecules of the type C_{60}, C_{70}, C_{76}, C_{84}, in which all atoms of carbon are on a spherical or spheroid surface. In these molecules the atoms of carbon are located in vertexes of regular hexagons or pentagons, which cover a surface of sphere or spheroid. We will start from a brief history of the molecule C_{60}. The molecule of the type C_{60} plays a special role among fullerenes. This molecule is characterized by the greatest symmetry and as consequence by the greatest stability. By its shape the molecule C_{60} reminds a typical white and black soccer football, the Telstar (football) of 1970 (Fig. 42), having a structure of the truncated regular icosahedron.

Figure 42. The Telstar (football) of 1970

The atoms of carbon in this molecule are located on the spherical surface in the vertexes of the 20 regular hexagons and 12 regular pentagons; here each hexagon is connected with three hexagons and three pentagons, and each pentagon is connected with hexagons (Fig. 43). The most striking property of the C_{60} molecule is its high symmetry. There are 120 symmetry operations, like rotations around an axis or reflections in a plane, which map the molecule onto itself. This makes C_{60} the molecule with the largest number of symmetry operations, the *most symmetric molecule*.

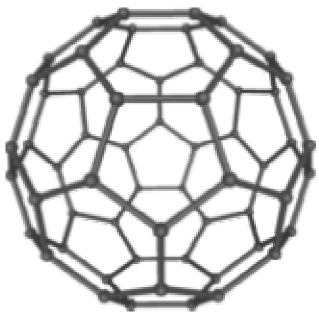


Figure 43. A structure of the molecule C_{60}

It should come as no surprise that a shape as symmetric and beautiful as that of the C60 molecule, has occupied many artists and mathematicians during many centuries. The truncated icosahedron was already known to Archimedes. The oldest known picture of the soccerball-shape seems to be a drawing found in the Vatican library. This picture is from a book of the painter and mathematician Piero della Francesca and dates from the 1480s. Johannes Kepler coined the name truncated icosahedron for this shape.

Fullerenes in essence are the "man-made" structures following from fundamental physical researches. New forms of the element carbon,called fullerenes, in which the atoms are arranged in closed shells, was discovered in 1985 by *Robert F. Curl*, *Harold W. Kroto* and *Richard E. Smalley*. They realized early that it should be possible to enclose metal atoms in the fullerene cages and thereby completely change the properties of the metal. Fullerenes are formed when vaporised carbon condenses in an atmosphere of inert gas. The gaseous carbon is obtained e.g. by directing an intense pulse of laser light at a carbon surface. The released carbon atoms are mixed with a stream of helium gas and combine to form clusters of some few up to hundreds of atoms. The gas is then led into a vacuum chamber where it expands and is cooled to some degrees above absolute zero. The carbon clusters can then be analysed with mass spectrometry.

Curl, Kroto and Smalley performed this experiment in 1985. By fine-tuning the experiment they were able in particular to produce clusters with 60 carbon atoms and clusters with 70. Clusters of 60 carbon atoms, C_{60}, were the most abundant. They found high stability in C60, which is a molecular structure of great symmetry. They suggested that C60 could be a "truncated icosahedron". The pattern of a European football has exactly this structure, as and the geodetic dome designed by the American architect R. Buckminster Fuller. The researchers named the newly-discovered structure *buckminsterfullerene* after him. In 1996 they win Nobel Prize for this discovery.

Fullerenes possess unusual chemical and physical properties. So, at high pressure C60 becomes firm, as diamond. Its molecules form the crystal structure as though consisting of ideally smooth spheres, freely rotating in a cubic lattice. Owing to this property C60 can be used as firm greasing. Fullerenes possess also magnetic and superconducting properties.

6.3. Resonance theory of the Solar system. The searches of regularities in the planet arrangement and rotations around of the Sun are continued up to now. More recently (1978) the Russian astronomer Butusov calculated the mean planet's cycle times and compared them to the "golden" geometrical progression [23]. Butusov found that the ratios of the adjacent planets cycle times around of the Sun are equal or to the golden proportion 1,618, or to its square 2,618. Also Butusov discovered the following regularity:

> ***"The frequencies and the frequencies differences of planet's circulations form the frequency spectrum with the interval, equal to τ, i.e. the spectrum constructed on the "golden section"! In other words, the spectrum of gravitational and acoustic disturbances created by the planets represents by itself the consonance chord, the most perfect chord from the acoustic point of view ...***
>
> ***It seems rather surprising, that Kepler who wrote about the golden section and studied the problem of the Universe harmony cannot discover this regularity!***

Finishing this paragraph, we can make a conclusion that the statements of Pythagoreans and Kepler about the "music of spheres" really correspond to the actual facts, instead of only symbolical".

What is a cause of such "strange" behavior of planets, which motions are subjected to the "golden section"? Recently the relevant scientific discovery was made in the mechanics field. This discovery certified an existence of the new natural phenomenon, the synchronization of the rotated bodies based on the resonance phenomenon, which results to the fact that the definite phase relations are established between speeds of the rotated bodies.

There is a question: whether is the resonance phenomenon the "main conductor of the planet's cosmic ensemble"? The mathematical calculations given by Butusov demonstrate that the solutions of the algebraic equations describing the resonance phenomena between planets are reduced really to either the golden proportion or its square.

A significance of the resonance phenomenon in the Nature and Engineering is difficult for overestimating. In the Engineering the resonance phenomenon is taken into consideration at calculation of machine designs and other engineering facilities. The resonance phenomenon underlies the harmonic combination of tones in musical works. The miscellaneous biorhythms of the human body, functioning its separate organs, for example, heart palpation are submitted to the resonance phenomenon. In the last years the hypothesis, which explains the nature of harmonic proportions, in particular, the golden proportion, by the resonance phenomenon, appeared. The harmonic proportions of the alive organisms ensuring their life-activity also are caused by the resonance phenomenon. According to this hypothesis the resonance as the invisible conductor adjusts systems, integrates them in the harmonic whole, subordinates to the overall life rhythm. Without the resonance there is no melody, there is no charm of musical works affecting to our hearts.

The planet arrangement and circulations regularities given above and based on the golden proportion are rather convincing. It is possible to expect, that the laws of development of different natural systems, the laws of their growth are not so miscellaneous and are tracked in the most different systems. In it the unity of the nature also shows. The idea of such unity based on development of the same regularities in heterogeneous natural phenomena kept its actuality from Pythagoras to our days.

6.4. Law of spiral symmetry transformation. As is well known from biology a relative arrangement of very different sprouts arising in the cones of shoots is characterized by the "spiral symmetry". It is well known that the process of the collective fruit growing is accompanied at the certain stage by a modification of the spiral symmetry order. As this takes place the modification is strictly regular and corresponds to the general rule of constructing the recurrent number sequences generated by Fibonacci recursive formula (7). In the case of Fibonacci's phyllotaxis a progress of symmetry order is presented through the sequence:

$$1:2 \Rightarrow 2:3 \Rightarrow 3:5 \Rightarrow 5:8 \Rightarrow 8:13 \Rightarrow 13:21 \Rightarrow \ldots \quad (45)$$

A change of the symmetry orders of phyllotaxis objects in accordance with (45) is called the *dynamic symmetry*. A remarkable illustration of the dynamic symmetry is given by the fact of a regular difference of the spiral symmetry orders in the sunflower heads located on different levels of one and the same stem.

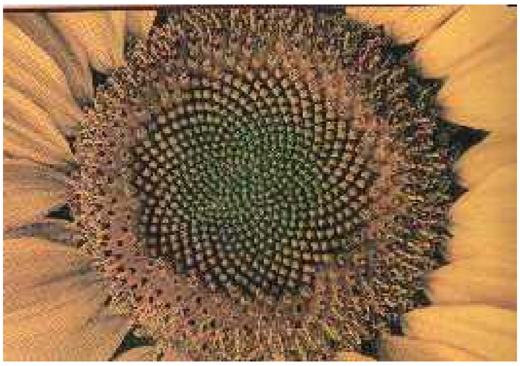


Figure 44. A head of sunflower

The spiral numbers in the sunflower discs are in direct dependence on their "age", i.e. the "older" disc corresponds to the bigger Fibonacci numbers. Most often the symmetry order of discs belonging to the same stem is characterized by the ratios of the Fibonacci numbers: 13:21, 21:34, 34:55, 55:89. It is these all data that constitute the essence of the universally known *"puzzle of phyllotaxis"*. A number of scientists, investigating this problem, assume that the phenomenon of phyllotaxis is of fundamental interdisciplinary importance. In the opinion of the famous Russian scientist V. Vernadsky, the problem of the biology symmetry is the key problem of biological science.

A new solution of the *"puzzle of phyllotaxis"* was given recently by the Ukrainian scientists O. Bodnar in his book "Golden Section and Non-Euclidean Geometry in Nature and Art" (1994) [24].

Bodnar's book "The Golden Section and non-Euclidean geometry in the Nature and Art" (1994)

Using the Fibonacci hyperbolic functions (23) Bodnar proved that Fibonacci numbers arising at the surface of the phyllotaxis objects are a consequence of hyperbolic character of growth processes of the phyllotaxis objects. But the living nature uses the Fibonacci hyperbolic functions for construction of its objects and this fundamental fact is confirmed by the "phyllotaxis laws" based on Fibonacci numbers! Bodnar's discovery is a brilliant confirmation of Fibonacci character of living Nature surrounded us!

6.5. Structural harmony of systems. It is clear that increasing the interest in problem of harmony and golden section in modern science has found its reflection in modern philosophy in form of new original philosophical concepts. The Byelorussian philosopher Eduard Soroko who advanced in the 80^{th} the highly interesting concept of "structural harmony of systems" developed one of similar concepts. This concept is called the *"Law of Structural Harmony of Systems"* and can be rightfully considered as one of the greatest philosophical achievements of the 20th century.

Eduard Soroko

Soroko's main idea is considering all real systems since "dialectical point of view". As is well known any natural object can be presented as the dialectical unity of the two opposite sides *A* and *B*. This dialectical connection may be expressed in the following form:

$$A + B = U \text{ (universum).} \quad (46)$$

The equality of (46) is the most general expression of so-called *conservation law*. Here *A* and *B* are some distinctions inside of the unity, logically non-crossing classes or substratum states of any whole. There exists the only condition that *A* and *B* should be measured with the same measure and be by members of the ratio underlying inside the unity. Probability and improbability of events, mass and energy, nucleus of atom and its envelope, substance and field, anode and cathode, animals and plants, spirit and material origin in the value system, profit and cost price, etc. are examples of (46).

The partial case of (46) is the *"law of information conservation"*:

$$I + H = \log N, \tag{47}$$

where I is a quantity of information and H is an entropy of the system having N states.

In process of self-organization every system strives to some "harmonies" state when some stable proportion arises between parts A and B in (46) (or I and H in (47)). Studying this proportion Soroko proved that one of the golden p-proportion τ_p being the root of the algebraic equation of (35) is the proportion of "harmonies" state of self-organized system. In accordance with Soroko's opinion, the golden p-proportions τ_p expresses the "Law of the structural harmony of systems", which according to Soroko sounds as of the following:

"Generalized golden sections are invariants, which allow natural systems in process of their self-organization to find harmonious structure, stationary regime of their existence, structural and functional stability".

Thus, basic Soroko's achievement consists of the fact that he broadened essentially a number of "harmonies" proportions of self-organized systems by using a concept of the generalized golden sections or golden p-sections. It is clear that practical usage of the "Law of Structural Harmony of Systems" can give essential advantage for solution of many technological, economical, ecological and other problems, in particular, can promote to improvement of technology of structural and complicated products, to monitoring of biosphere etc.

6.6. "Fibonacci's resonances" of Genetic Code. In 1990 Jean-Claude Perez, an employee of IBM, made a rather unexpected discovery in the field of genetic code [25]. He discovered the mathematical law controlling the self-organizing of the *T, C, A,* and *G* bases inside of DNA. He found that the consecutive sets of DNA nucleotides are organized in frames of the distant order called "Resonances." Here "Resonance" represents the special proportion ensuring division of DNA parts pursuant to the three neighboring Fibonacci numbers (1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, …), for example 55-34-21, 89-55-34, etc.

Let us consider the DNA molecule of insulin, one of the simplest DNA molecules. It consists of two circuits, α- and β-circuits. For the β-circuit the sequence of triplets has the following form:

ATG-TTG-GTC-AAT-CAG-CAC-CTT-TGT-GGT-TCT-CAC-CTC-GTT-GAA-GCT-
TG-TAC-CTT-GTT-TGC-GGT-GAA-CGT-GGT-TTC-TTC-TAC-ACT-CCT-AAG-ACT

If we mark all T-bases in red and the others by yellow and count the number of all bases (3×30=90) we have get the following outcome: 34 T-bases and 56 other bases. Thus we have the following proportion between the bases: 90-56-34, very close to Fibonacci's "resonance": 89-55-34. It means that Jean-Claude Perez's law is fulfilled for the insulin DNA molecule with sufficient accuracy for practice.

This discovery is doubtless one of the outstanding discoveries in genetic coding field. In the opinion of Jean-Claude Perez, the SUPRA-code of DNA indicates the highest level of self-organizing of nucleotides in DNA according to the principle of the golden section. This surprising discovery allows a further connection between music, poetry, market fluctuations (Elliott Wave theory) and genetic code. It is clear that the harmony of Chopin's etudes and Pushkin's poetry are similar to the harmony of the genetic code, in which Fibonacci's resonances are seen both in the DNA molecule and in all its parts.

6.7. Some useful analogies between Fibonacci and genetic codes. Among biological concepts, well formalized and having a level of the general scientific importance, the genetic code takes a special place. Discovering the well-known fact of striking simplicity of basic principles of genetic code falls into the major modern discoveries of mankind. This simplicity consists of the fact that the inheritable information is encoded by the texts from the three-alphabetic words - triplets or codonums compounded on the basis of the alphabet consisted of four characters being the nitrogen bases: **A** (adenine), **C** (cytosine), **G** (guanine), **T** (thiamine). The given recording system is essentially unified for all boundless set of miscellaneous alive organisms and is called genetic code [26].

It is believed that by using three-alphabetic triplets or codonums we can code 20 amino acids. Professor Sergey Petoukhov attracts author's attention that in addition to 20 amino asides there exists one more item, so-called *stop-codonum* (sign of the punctuation) encoded by triplets. Then there exist $4^3=64$ different combinations from four on three nitrogen bases used for coding 21 items (20 amino acids and one stop-codonum). In this connection some of 21 items are encoded at once by several triplets. It is called as a *degeneracy of the genetic code*. Finding of conformity between triplets and amino acids (or signs of the punctuation) is customary treated as *decryption of genetic code*.

Let us consider now the 6-digit Fibonacci code used Fibonacci numbers 1, 1, 2, 3, 5, 8 as digit weights [12, 13, 14]:

$$N = a_6\times 8 + a_5\times 5 + a_4\times 3 + a_3\times 2 + a_2\times 1 + a_1\times 1 \qquad (48)$$

Studying Fibonacci representation of (48) one can find the following surprising analogies between the 6-digit Fibonacci code (48) and genetic code:

(1) ***The first analogy.*** For representation of numbers the 6-digit binary Fibonacci code (48) uses $2^6 = 64$ binary code combinations from 000000 up to 111111 that coincides with the number of triplets of genetic code: $4^3 = 64$.

(2) ***The second analogy***. Using the 6-digit Fibonacci code (48) it is possible to represent 21 integers starting since the number of 0 encoded by the 6-digit binary combination 00000 and ending with the maximum number 20 encoded by the 6-digit code combination 111111. Note that using triplet's coding we can represent also 21 objects including 20 amino acids and one additional object used for coding the stop-codonum (the sign of the punctuation).

(3) ***The third analogy***. The main feature of Fibonacci code is a multiplicity of number representation. Except of the minimum number of 0 and the maximum number of 20, which have in Fibonacci code alone code representations (accordingly 000000 and 111111), all remaining numbers from 1 up to 19 have in Fibonacci code multiple representations, that is, they use not less than two code combinations for their representation. It is necessary to note that in genetic code property of multiplicity of representation also is used and it is called *"degeneracy"* of genetic code.

Thus, between the 6-digit Fibonacci code and genetic code based on triplet's representation of amino acids [26] there are rather interesting analogies, which allow to select Fibonacci code [12, 13, 14] as a special class of redundant codes among other ways of redundant coding. One may express assumption that just a study of Fibonacci code and Fibonacci arithmetic

[12, 13, 14, 16, 19] can promote finding new features of genetic code. It is possible to assume that similar analogies can become rather useful at design of the DNA-based bio-computers.

6.8. The "golden" genomatrices by Sergey Petoukhov. The Russian scientist Sergey Petoukhov has made recently scientific discovery in the genetics, which turn over our representations about the genetic code []. We can formulate this discovery as follows:

Petoukhov's discovery. Let *A* (adenine), *C* (cytosine), (*G*) (guanine), *U* (uracil) be nitrogen bases ("letters") of the genetic alphabet, which form the initial symbolic matrix $P=\begin{pmatrix} C & A \\ U & G \end{pmatrix}$**. Let $P^{(n)}$ be a symbolic genomatrix formed by means of tensor (Kronecker) raising the initial symbolic matrix *P* to *n*-th power and let polyplets, which are formed from the "letters" *A, C, G* and *U,* be elements of the symbolic matrix $P^{(n)}$. If we form now from the symbolic genomatrix $P^{(n)}$ some numerical genomatrix $P^{(n)}_{мульт}$ by means of replacing every polyplet of the matrix $P^{(n)}$ by a number equal to the product of the numbers of hydrogen relations of its nitrogen bases according to the rule: *A*=*U* = 2 and *C*=*G*=3 and if we form from the symbolic genomatrix $P^{(n)}$ some "golden" genomatrix $\Phi^{(n)}$ by means of replacing every polyplet of the symbolic matrix $P^{(n)}$ by the product of the following values of its "letters" according to the rule: $C=G=\tau$, $A=U=\tau^{-1}$, where $\tau=\frac{1+\sqrt{5}}{2}$ is the golden proportion, then there is the following fundamental relation between the numerical genomatrix $P^{(n)}_{мульт}$ and the "golden" genomatrix $\Phi^{(n)}$:**

$$(\Phi^{(n)})^2 = P^{(n)}_{мульт}.$$

We can consider different examples of Petoukhov's "golden" genomatrices. Represent the letters *A, C, G* and *U* of the genetic code in the form of the following 2×2-matrix:

$$P=\begin{pmatrix} C & A \\ U & G \end{pmatrix} \tag{48}$$

If we substitute the letters *A, C, G* and *U* by the numbers equal to numbers of hydrogen relations of its nitrogen bases according to the rule: *A*=*U* = 2 and *C*=*G*=3, we will get the following numerical genomatrix:

$$P^{(1)}_{mult}=\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} \tag{49}$$

If we substitute the letters *A, C, G* and *U* of the matrix (48) according to the rule: $C=G=\tau$, $A=U=\tau^{-1}$ ($\tau=\frac{1+\sqrt{5}}{2}$ is the golden ratio), we will get the "golden" genomatrix

$$\Phi^{(1)}=\begin{pmatrix} \tau & \tau^{-1} \\ \tau^{-1} & \tau \end{pmatrix} \tag{50}$$

which is connected with the numerical genomatrix (49) as follows:

$$(\Phi^{(1)})^2=\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} \tag{51}$$

This means that the numerical genomatrix (49) is equal to the square of the "golden" genomatrix (50).

This regularity is general for all possible genomatrices, including the genomatrix of triplets:

Symbolic genomatrix

$$P^{(3)} = P \otimes P \otimes P = \begin{pmatrix} CCC & CCA & CAC & CAA & ACC & ACA & AAC & AAA \\ CCU & CCG & CAU & CAG & ACU & ACG & AAU & AAG \\ CUC & CUA & CGC & CGA & AUC & AUA & AGC & AGA \\ CUU & CUG & CGU & CGG & AUU & AUG & AGU & AGG \\ UCC & UCA & UAC & UAA & GCC & GCA & GAC & GAA \\ UCU & UCG & UAU & UAG & GCU & GCG & GAU & GAG \\ UUC & UUA & UGC & UGA & GUC & GUA & GGC & GGA \\ UUU & UUG & UGU & UGG & GUU & GUG & GGU & GGG \end{pmatrix} \quad (52)$$

Numerical genomatric

$P^{(3)}_{mult} =$ (53)

									∑
	27	18	18	12	18	12	12	8	125
	18	27	12	18	12	18	8	12	125
	18	12	27	18	12	8	18	12	125
	12	18	18	27	8	12	12	18	125
	18	12	12	8	27	18	18	12	125
	12	18	8	12	18	27	12	18	125
	12	8	18	12	18	12	27	18	125
	8	12	12	18	12	18	18	27	125
∑	125	125	125	125	125	125	125	125	1000

"Golden" genomatrix

$$\Phi^{(3)} = \begin{pmatrix} \tau^{3} & \tau^{1} & \tau^{1} & \tau^{-1} & \tau^{1} & \tau^{-1} & \tau^{-1} & \tau^{-3} \\ \tau^{1} & \tau^{3} & \tau^{-1} & \tau^{1} & \tau^{-1} & \tau^{1} & \tau^{-3} & \tau^{-1} \\ \tau^{1} & \tau^{-1} & \tau^{3} & \tau^{1} & \tau^{-1} & \tau^{-3} & \tau^{1} & \tau^{-1} \\ \tau^{-1} & \tau^{1} & \tau^{1} & \tau^{3} & \tau^{-3} & \tau^{-1} & \tau^{-1} & \tau^{1} \\ \tau^{1} & \tau^{-1} & \tau^{-1} & \tau^{-3} & \tau^{3} & \tau^{1} & \tau^{1} & \tau^{-1} \\ \tau^{-1} & \tau^{1} & \tau^{-3} & \tau^{-1} & \tau^{1} & \tau^{3} & \tau^{-1} & \tau^{1} \\ \tau^{-1} & \tau^{-3} & \tau^{1} & \tau^{-1} & \tau^{1} & \tau^{-1} & \tau^{3} & \tau^{1} \\ \tau^{-3} & \tau^{-1} & \tau^{-1} & \tau^{1} & \tau^{-1} & \tau^{1} & \tau^{1} & \tau^{3} \end{pmatrix} \quad (54)$$

If we square the matrix (54) then after corresponding transformation we will get:

$$\left(\Phi^{(3)}\right)^2 = P^{(3)}_{мульт}, \quad (55)$$

Petoukhov's discovery [25] shows a fundamental role of the "golden proportion" in genetic code. This discovery gives evidence that the golden section underlies a Living Nature! Now still it is difficult to estimate in full measure a revolutionary character of Petoukhov's discovery for the development of modern science. It is clear that this discovery is a scientific result of the same importance, as well as discovery of genetic code!

6.9. Fibonacci numbers and Hilbert's Tenth Problem. To explain an essence of this famous mathematical problem we should return more then 17^{th} centuries back to the famous Greek mathematician Diophantus who lived in the 3^{rd} century A.C. Diophantus became famous for searching solutions of algebraic equations in the domain of integer numbers. Such algebraic equations are called Diophantine one's. Let us consider, for example, the simplest Diophantine equation $x^2+y^2=z^2$, connecting sides x, y, z of a rectangular triangle. The natural numbers x, y and z, being solution of this equation, are called *"Pythagorean triples"*. The numbers of 3, 4, 5 are those, because $3^2+4^2=5^2$. We already mentioned in our Museum, that the triangle with such sides was called "sacred" or "Egyptian" and was put by the ancient Egyptians in the basis of Chefren's Pyramid.

However, we can consider the following Diophantine equation:

$$x^n + y^n = z^n. \tag{56}$$

This equation was written by the famous 16^{th} century mathematician Fermat on the margins of Diophantus' book, where he wrote the following inscription: *"... It is impossible to decompose neither cube on two cubes, nor biquadrate on two biquadrates and in general any degree more then square on two degrees with the same exponent. I discovered to this indeed wonderful proof, but these margins are too narrow"*. In other words, according Fermat's opinion the equation (56) for the case $n>2$ has not solutions in natural numbers. Many great mathematicians (Euler, Legendre, Kummer) participated in the proof of *"Fermat's Last Theorem"* and resulted the rather complicated proofs of its true for particular cases. In this connection Fermat's statement that his proof of this theorem had not located on the margins seems simply incredible. Fermat's 350 years old hypothesis that none of these equations has non-zero integer solutions, was 100% proved as late as in September 19, 1994 (the last step was made by the English mathematician Andrew Wiles).

In general Diophantine equations have the following form:

$$D(x_1, x_2, x_3, \ldots) = 0, \tag{57}$$

where D is a polynomial with integer coefficients.

Then there arises a problem for recognizing the solvability of Diophantine equations for general case of (57).

In the summer of 1900 mathematicians met on the Second International Congress in Paris. In the lecture *"Mathematical Problems"* delivered by the famous German mathematician David Hilbert (1862-1943) 23 major mathematical problems were formulated. Hilbert's Tenth Problem is called **"Determining the solvability of a Diophantus equation"**. Many outstanding World mathematicians, in particular, the American mathematicians Martin Davis and Julia Robinson, tried to solve this problem during many decades. The last drop in this solution was made in 1970 by the Russian mathematician Yuri Matiyasevich. We have not possibility to explain in detail an essence of this problem and different approaches to its solution offered by Martin Davis, Julia Robinson, and Yuri Matiyasevich because it demands on special mathematical knowledge. But it is the most interesting for us that this problem was solved by Yuri Matiyasevich by means of application of Fibonacci numbers! However let us give a world to Yuri Matiyasevich himself:

"Thanks to my previous work, I realized the importance of Fibonacci numbers for Hilbert's tenth problem. That is why during summer of 1969 I was reading with great interest the third augmented edition of a popular book on Fibonacci numbers written by N.N. Vorob'ev from Leningrad. It seems incredible that in the 20th century one can still find something new about the numbers introduced by Fibonacci in the 13th century in connection with multiplying rabbits.

However, the new edition of the book contained, besides traditional stuff, some original results of the author. In fact, Vorob'ev had obtained them a quarter of a century earlier but he never published anything before. His results attracted my attention at once but I was not able to use them immediately for constructing a Diophantus representation of a relation of exponential growth".

A history of this mathematical discovery is the brightest example how international collaboration of the outstanding World mathematicians can promote to scientific progress. The American mathematicians Martin Davis and Julia Robinson and the Russian mathematician Yuri Matiyasevich are the most well-known World mathematician in the area of the Tenth Hilbert's Problem. Below we can see a unique photo of these outstanding mathematicians. The photo was taken in Calgary at the end of 1982 when Yuri Matiyasevich spent three months in Canada as participant of a scientific exchange program between the Steklov Institute of Mathematics and Queen's University at Kingston, Ontario. At that time Julia Robinson was very much occupied with her new duties as President of the American Mathematical Society but she was able to visit Calgary to meet Yuri Matiyasevich. Martin Davis also came to Calgary for a few days.

From left to right: Martin Davis, Julia Robinson, Yuri Matiyasevich (Calgary, 1982).

We would like to conclude this part of our article devoted to the history of the outstanding 20th century mathematical discovery by the quote from Julia Robinson's letter to Yuri Matiyasevich:

> "*Actually I am very pleased that working together (thousands miles apart) we are obviously making more progress than either one of us could alone".*

Notice that this sentence was written in period of the "Cool War" between USA and Soviet Union. And these nice words show that friendship and scientific collaboration do not depend on political situation because aims of Science were always higher and cleaner of political speculations.

7. Harmonic Education

It is clear that awareness of the Golden Section and its applications in Nature, Music, Art, and Science could enrich the life and work of scholars in a variety of professions. A question arises instinctively: Why does this information not be taught in secondary school? Educators seem unable to answer this question.

The reason for this may be in tradition. Traditionally the classic pedagogy treats the golden section with some prejudice as it relates to astrology and "esoteric" sciences. But the connections we have enumerated here—phyllotaxis, Shechtman's quasi-crystals, Perez's discovery, and others—show that the classic "materialistic" science is moving now to the embraces of the "esoteric" science!

The astronomer Johannes Kepler once said: *"There are two treasures in Geometry: Pythagorean Theorem and division of a line segment in extreme and mean ratio ("golden ratio"). The former can be compared to value of gold; the latter can be named as a gemstone."* But every pupil knows the Pythagorean Theorem; but sometimes even famous scientists and teachers have a dim knowledge of the Golden Section. The first step in improving education is integrating the Golden Section into the school mathematics (geometry, algebra, number theory); the second step is to integrate it into the natural and social science disciplines (physics, chemistry, astronomy, botany, biology, anatomy, psychology, sociology, economics, and computer science). We should also teach the principles of the Golden Section in art classes (golden rectangle, golden spiral, pentagon, Fibonacci lattices, etc.) and include examples of their usage in architecture, painting, sculpture, music, and poetry. Thus our teaching would promote a new scientific outlook based on principles of harmony and the golden section.

8. Architectural Realization of the Museum of Harmony and the Golden Section

The author's main goal is the physical realization of the Museum of Harmony and the Golden Section as a grandiose architectural temple in which works of nature, science, and art based on the golden section would be collected. We are therefore asking outstanding architects to participate in this project, and we ask businessmen, creative persons, sports figures, artists, actors, musicians, and anyone else who could support the project financially to become sponsors and participants in this undertaking. In the meantime, we welcome our readers to visit our "Virtual Museum of Harmony and Golden Section" at http://www.goldenmuseum.com/ [1].

Recently the author has received very interesting letter from far Australia. And the author would like to finish his article by the words of the Australian journalist Dimity Torbett:

"I am writing from Sydney, Australia, to thank you for your brilliant idea for a Museum of Harmony and the Golden Section. As soon as I saw it on the internet, I thought that such museums should be built wherever possible in the world - for instance, in Sydney Harbour there is an island once used for industry and now just lying idle that would be a perfect location. And wherever such museums were built, they could bring together the talents of the nation's scientists, mathematicians, artists, architects etc. Every museum would be different, reflecting the

individual approaches of each country. They would stimulate not only people's minds but also the economy by creating jobs, new industries, tourism etc.

Soon after finding your website, there came the events of September 11 and it struck me that if anything were to be built on the site of the wrecked towers, it should be the first Museum of Harmony - for what could make a finer, more inspiring contribution to all mankind or do more to help to expand people's consciousness and open their eyes to an understanding of the inherent beauty and harmony of our world?"

For development of this idea the initiative grope from the representatives of USA, Ukraine and Australia was organized. The group elaborated very interesting proposal about architectural realization of the Museum of Harmony and the Golden Section (see http://www.goldennumber.net/NYHarmony.htm). Of course, every country and every continent has a chance to realize this idea but the United State of America might become the first country where this unique Museum could be realized.

References

[1] Stakhov A.P., Sluchenkova A. A. Web site "Museum of Harmony and Golden Section", 2001 http://www.goldenmuseum.com/

[2] Shmelev I.P. *Phenomenon of the Ancient Egypt.* Minsk, Publisher "Lotaz", 1993 [In Russian].

[3] Vasutinski N.A. *The Golden Proportion.* Moscow, Publisher "Molodaja Gvardija", 1990 [In Russian].

[4] Borodin A.I., Bugai A.S. *Biographic dictionary of the persons in the field of mathematics"*, Kiev, Publisher "Radjanska Shkola", 1979 [In Russian].

[5] Vorobev N.N. *Fibonacci numbers.* Moscow, Publisher "Nauka", 1961 [In Russian].

[6] Hoggat, V., Jr., *Fibonacci and Lucas Numbers*, Houghton-Mifflin, Palo Alto, California, 1969.

[7] Luca Pacioli. *Treatise on Counting and Records.* Moscow, Publisher "Finance and Statistics", 1983 [In Russian].

[8] Meisner Gary. WEB site "The Stock Market and the Fibonacci Series" http://goldennumber.net/stocks.htm

[9] Tsvetkov V.D. *Heart, Golden Section, Symmetry.* Pushcino, Publisher "Puschino nauchny zentr", 1997 [In Russian].

[10] Sabaneev Z. "Shopin's Etudes in the lighting of the golden section". *Art Journal "Iskusstvo",* Volume 2, 1925, pp. 132-135 [In Russian].

[11] F.R. Paturi. *The Plants as Ingenious Engineers of Nature.* Moscow, Publisher "Progress", 1982 [In Russian].

[12] Stakhov A.P. *Introduction into Algorithmic Measurement Theory.* Moscow, Publisher "Soviet Radio", 1977 [In Russian].

[13] Stakhov A.P. *Algorithmic Measurement Theory.* Moscow, Publisher "Znanie", Series "Mathematics & Cybernetics", No 6, 1979 [In Russian].

[14] Stakhov A.P. *Codes of the Golden Proportion.* Moscow, Publisher "Radio and Communications", 1984 [In Russian].

[15] Bergman G. A number system with irrational base. *Math. Mag.,* 31, 1957, pp. 98-119.

[16] Stakhov A.P. Brousentsov's Ternary Principle, Bergman's Number System and Ternary Mirror-symmetrical Arithmetic. *The Computer Journal,* Vol. 45, No 2, 2002, 221-236.

[17] Stakhov A.P. A generalization of the Fibonacci *Q*-matrix. *Reports of the National Academy of Sciences of Ukraine,* No 9, 1999.

[18] Gould H.W. A history of the Fibonacci *Q*-matrix and a higher-dimensional problem. *The Fibonacci Quarterly,* 19, No 3, 1981, pp. 250-257.
[19] Stakhov A.P., Massingua V., Sluchenkova A.A.. *Introduction into Fibonacci Coding and Cryptography.* Kharkov, Publisher "Osnova" of the Kharkov State University, 1999.
[20] Stakhov A.P., Tkachenko I.S. Hyperbolic Fibonacci Trigonometry. *Reports of Academy of Sciences of Ukraine*, Vol. 208, No 7, 1993.
[21] Stakhov A.P. The Golden Section and Modern Harmony Mathematics. *Applications of Fibonacci Numbers*, 1998, V. 7, pp. 323-399.
[22] Stakhov A.P. The Golden Section and Modern Harmony Mathematics. *Boletin de Informatica.* The Golden Section: Theory and Applications, 1999, No 9-10, pp. 3-24.
[23] Butusov K.P. The Golden Section in the Solar System, *Problemy issledovania vselennoy*, V. 7, 1978, pp. 475 - 500 [In Russian].
[24] Bodnar O.Y. *Golden Section and Non-Euclidean geometry in Nature and Art.* Lvov, Publisher "Svit", 1994 [In Russian].
[25] Perez Jean-Claude. WEB site "The DNA SUPRA code", 2000
http://www.genum.com/dna_supracode/dna_supracode.htm
[26] Petoukhov S.V. *The Biperiodical Table of Genetic Code and Number of Protons".* Moscow, Publisher "Molodegny knigny zentr", 2001.
[27] Petoukhov SV. Metaphysical aspects of the matrix analysis of the genetic code and the golden section. Metaphysics. Century XXI, Moscow, BINOM, 2006: 216-250.